The Eshelby Tensors in a Finite Spherical Domain—Part I: Theoretical Formulations

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This work is concerned with the precise characterization of the elastic fields due to a spherical inclusion embedded within a spherical representative volume element (RVE). The RVE is considered having finite size, with either a prescribed uniform displacement or a prescribed uniform traction boundary condition. Based on symmetry and group theoretic arguments, we identify that the Eshelby tensor for a spherical inclusion admits a unique decomposition, which we coin the "radial transversely isotropic tensor." Based on this notion, a novel solution procedure is presented to solve the resulting Fredholm type integral equations. By using this technique, exact and closed form solutions have been obtained for the elastic disturbance fields. In the solution two new tensors appear, which are termed the Dirichlet–Eshelby tensor and the Neumann–Eshelby tensor. In contrast to the classical Eshelby tensor they both are position dependent and contain information about the boundary condition of the RVE as well as the volume fraction of the inclusion. The new finite Eshelby tensors have far-reaching consequences in applications such as nanotechnology, homogenization theory of composite materials, and defects mechanics. [DOI: 10.1115/1.2711227]

1 Introduction

One of the corner stones of contemporary micromechanics and nanomechanics is Eshelby's inclusion theory [1–3]. Eshelby's ellipsoidal inclusion solution was obtained based on the assumption that an inclusion is embedded in unbounded ambient space. This is a good approximation if the size effect of the inclusion is negligible, i.e., the size of the inclusion is small compared to the size of the representative volume element. In engineering applications, the size of the representative volume element (RVE) is finite. Therefore, certain approximations have to be made in order to utilize Eshelby's classical solution in homogenization. This limitation becomes obvious, when size effects and interfacial boundary effects of a second phase in a composite, or the size effect and boundary effects of an inhomogeneity, become prominent issues, which is one of main focuses of the nanocomposite mechanics and materials, e.g., Refs. [4,5]. Today, there is a call for the solution of the inclusion problem in a finite domain.

Inclusion problems in a finite domain have been considered before, e.g., Refs. [6–9]. A common approach adopted is to first find the Green's function of Navier's equation for a finite domain, and then to find the solution of the corresponding inclusion problem. However, attempts based on this approach have been futile, we believe, because of the mathematical difficulties involved in obtaining a closed form solution of the finite Green's function. This is true even for a highly symmetrical spherical domain. In fact, the Green's function of Navier's equation for a finite spherical domain has not been found yet. To the best of the authors' knowledge, there has never been any exact, closed form solution of the inclusion problem in a finite domain published in the literature. A solution has been obtained by Luo and Weng [10], which coincides with our solution in a special case. Their solution, however, is not in closed form and lacks expressions for the Eshelby tensors.

In this paper, which is the first part of a series, we present the exact solution of the finite Eshelby tensors of a spherical inclusion embedded concentrically within a finite spherical RVE. The following section illustrates the two boundary value problems (BVPs) we are considering and their resulting integral equations. In Sec. 3 the notion of a transversely isotropic tensor is discussed, which is used in Sec. 4 to solve the two integral equations. Section 5 concludes this part. The Eshelby tensors derived in this paper have some profound consequences for both homogenization and the study of inhomogeneities in finite elastic solids. In the second part of this work, applications to homogenization of composites are discussed [11].

2 The Inclusion Problem

We consider Eshelby's homogeneous inclusion problem in a finite domain. Figure 1 shows a spherical inclusion Ω_I with radius a embedded at the center of a spherical representative volume element Ω with radius A. Consider two arbitrary points $\mathbf{x} \in \Omega$, $\mathbf{y} \in \Omega$, and let $\mathbf{r}=\mathbf{y}-\mathbf{x}$. Each vector $\mathbf{x}$, $\mathbf{y}$, $\mathbf{r}$ can be expressed as its length multiplied by a unit direction vector. We shall denote them as $\mathbf{x}=|\mathbf{x}|\bar{\mathbf{x}}$, $\mathbf{y}=|\mathbf{y}|\bar{\mathbf{y}}$ and $\mathbf{r}=r\bar{\mathbf{r}}$, with $r=|\mathbf{r}|$. Note that if $\mathbf{y} \in \partial\Omega$ we have $|\mathbf{y}|=A$ and $\bar{\mathbf{y}}=\mathbf{n}$, i.e., the direction of $\mathbf{y}$ is equal to the outward surface normal $\mathbf{n}$. Furthermore we define the ratios $\rho=a/|\mathbf{x}|$, $\rho_0=a/A$ and $t=|\mathbf{x}|/A=\rho_0/\rho$ to allow for a nondimensional description. Suppose that a constant eigenstrain field is prescribed inside the inclusion

$$\epsilon_{ij}^*(\mathbf{x}) = \begin{cases} \epsilon_{ij}^*, & \mathbf{x} \in \Omega_I \\ 0, & \mathbf{x} \in \Omega_E = \Omega/\Omega_I \end{cases} \tag{1}$$

The infinitesimal elastic strain equals the total strain subtracting the eigenstrain

$$e_{ij} = \epsilon_{ij} - \epsilon_{ij}^*$$

with

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

where $u_{i,j}$ denotes the spatial differentiation $\partial u_i/\partial x_j$. We assume that the RVE is a linear elastic medium, i.e.,

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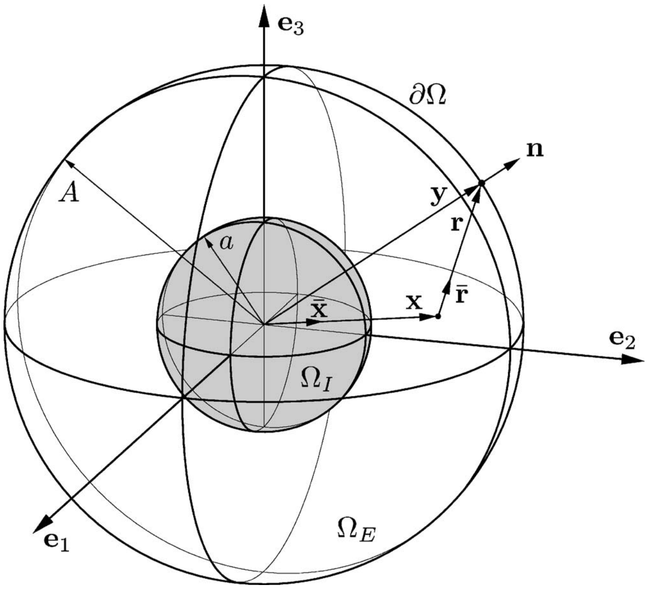

Fig. 1 A spherical representative element containing a spherical inclusion

$$\sigma_{ij} = \mathbb{C}_{ijk\ell} e_{k\ell}, \quad \forall\, \mathbf{x} \in \Omega \tag{3}$$

where σ_{ij} are the components of the Cauchy stress tensor, and $\mathbb{C}_{ijk\ell}$ is the elasticity tensor.

On the boundary of the RVE, two types of boundary conditions (BC), a prescribed displacement (Dirichlet) or a prescribed traction (Neumann) boundary condition, are considered

Dirichlet BC

$$u_i = \epsilon^0_{ij} x_j, \quad \forall\, \mathbf{x} \in \partial\Omega \tag{4}$$

Neumann BC

$$t_i = \sigma^0_{ij} n_j, \quad \forall\, \mathbf{x} \in \partial\Omega \tag{5}$$

where ε^0_{ij} and σ^0_{ij} are the background strain and stress fields. The elastic fields inside the RVE can be decomposed into the background field from the remote boundary loads and a disturbance field, which arises due to the presence of the inclusion. Thus the displacement and traction can be written as

$$u_i = u^0_i + u^d_i, \quad t_i = t^0_i + t^d_i \tag{6}$$

so that we obtain the following two homogeneous boundary conditions

Dirichlet BC

$$u^d_i = 0, \quad \forall\, \mathbf{x} \in \partial\Omega \tag{7}$$

Neumann BC

$$t^d_i = 0, \quad \forall\, \mathbf{x} \in \partial\Omega \tag{8}$$

for the disturbance fields. The solution for the background field depends on the macro problem. Here we are concerned with the solution of the disturbance fields. Considering the equilibrium equations $\sigma^d_{ji,j}=0$, we obtain either the Dirichlet–Eshelby BVP

$$\begin{aligned} \mathbb{C}_{ijk\ell} u^d_{k,\ell j}(\mathbf{x}) - \mathbb{C}_{ijk\ell} \epsilon^*_{k\ell,j}(\mathbf{x}) &= 0, \quad \forall\, \mathbf{x} \in \Omega \\ u^d_i(\mathbf{x}) &= 0, \quad \forall\, \mathbf{x} \in \partial\Omega \end{aligned} \tag{9}$$

or the Neumann–Eshelby BVP

$$\begin{aligned} \mathbb{C}_{ijk\ell} u^d_{k,\ell j}(\mathbf{x}) - \mathbb{C}_{ijk\ell} \epsilon^*_{k\ell,j}(\mathbf{x}) &= 0, \quad \forall\, \mathbf{x} \in \Omega \\ t^d_i(\mathbf{x}) = n_j \mathbb{C}_{ijk\ell} u^d_{k,\ell}(\mathbf{x}) &= 0, \quad \forall\, \mathbf{x} \in \partial\Omega \end{aligned} \tag{10}$$

Let us denote the Green's function, $G^\infty_{mi}(\mathbf{x}-\mathbf{y})$, as the solution of the following Navier's equation in unbounded space

$$\mathbb{C}_{ijk\ell} G^\infty_{mk,\ell j}(\mathbf{x}-\mathbf{y}) + \delta_{mi}\delta(\mathbf{x}-\mathbf{y}) = 0, \quad \forall\, \mathbf{x},\mathbf{y} \in \mathbb{R}^3, \quad i = 1,2,3 \tag{11}$$

For an isotropic linear elastic space, the Green's function is (e.g., Ref. [12])

$$G^\infty_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{(x_i-y_i)(x_i-y_i)}{r^3} + (3-4\nu)\frac{\delta_{ij}}{r}\right] \tag{12}$$

where $r=\sqrt{(x_i-y_i)(x_i-y_i)}$; μ is the shear modulus; and ν is Poisson's ratio. By using Somigliana's identity [12], the displacement field solution of BVPs Eqs. (9) and (10) may be expressed as

$$\begin{aligned} u^d_m(\mathbf{x}) = &-\int_\Omega \mathbb{C}_{ijk\ell} G^\infty_{im,j}(\mathbf{x}-\mathbf{y}) \epsilon^*_{k\ell}(\mathbf{y}) d\Omega_y \\ &+ \int_{\partial\Omega} \mathbb{C}_{ijk\ell} u^d_{k,\ell}(\mathbf{y}) G^\infty_{im}(\mathbf{x}-\mathbf{y}) n_j(\mathbf{y}) dS_y \\ &+ \int_{\partial\Omega} \mathbb{C}_{ijk\ell} u^d_k(\mathbf{y}) G^\infty_{im,j}(\mathbf{x}-\mathbf{y}) n_\ell(\mathbf{y}) dS_y \end{aligned} \tag{13}$$

where we have denoted $G^\infty_{im,j} := \partial G^\infty_{im}/\partial x_j = -\partial G^\infty_{im}/\partial y_j$. For the Dirichlet–Eshelby problem, this integral equation becomes

$$\begin{aligned} u^d_m(\mathbf{x}) = &-\int_\Omega \mathbb{C}_{ijk\ell} G^\infty_{im,j}(\mathbf{x}-\mathbf{y}) \epsilon^*_{k\ell}(\mathbf{y}) d\Omega_y \\ &+ \int_{\partial\Omega} \mathbb{C}_{ijk\ell} u^d_{k,\ell}(\mathbf{y}) G^\infty_{im}(\mathbf{x}-\mathbf{y}) n_j(\mathbf{y}) dS_y \end{aligned} \tag{14}$$

and for the Neumann–Eshelby problem, Eq. (13) reduces to

$$\begin{aligned} u^d_m(\mathbf{x}) = &-\int_\Omega \mathbb{C}_{ijk\ell} G^\infty_{im,j}(\mathbf{x}-\mathbf{y}) \epsilon^*_{k\ell}(\mathbf{y}) d\Omega_y \\ &+ \int_{\partial\Omega} \mathbb{C}_{ijk\ell} u^d_k(\mathbf{y}) G^\infty_{im,j}(\mathbf{x}-\mathbf{y}) n_\ell(\mathbf{y}) dS_y \end{aligned} \tag{15}$$

In case of the Dirichlet–Eshelby BVP, the disturbance strain field follows from the displacement Eq. (14) as

$$\begin{aligned} \epsilon^d_{ij}(\mathbf{x}) = &-\frac{1}{2}\epsilon^*_{mn}\int_{\Omega_I} \mathbb{C}_{k\ell mn}[G^\infty_{ki,\ell j}(\mathbf{x}-\mathbf{y}) + G^\infty_{kj,\ell i}(\mathbf{x}-\mathbf{y})] d\Omega_y \\ &+ \frac{1}{2}\int_{\partial\Omega} \mathbb{C}_{k\ell pq}\epsilon^d_{pq}(\mathbf{y})[G^\infty_{ki,j}(\mathbf{x}-\mathbf{y}) + G^\infty_{kj,i}(\mathbf{x}-\mathbf{y})] n_\ell(\mathbf{y}) dS_y \end{aligned} \tag{16}$$

For the Dirichlet–Eshelby BVP we solve Eq. (16) which is an integral equation for the unknown strain field ϵ^d_{ij}. In case of the Neumann–Eshelby BVP we can directly solve Eq. (15) which is an integral equation for the unknown displacement field u^d_i. In passing, we note that Eq. (16) becomes a hypersingular integral equation if $\mathbf{x} \in \partial\Omega$.

To illustrate our solution procedure, we re-examine the classical Eshelby tensors. For inclusion problems in unbounded space, the boundary term in Eqs. (14)–(16) drops out. One can then find the disturbance strain fields in terms of the Eshelby tensors [1,2],

$$\epsilon^d_{ij}(\mathbf{x}) = \mathbb{S}^{\cdot,\infty}_{ijk\ell}(\mathbf{x})\epsilon^*_{k\ell}, \quad \forall\, \mathbf{x} \in \mathbb{R}^3 \tag{17}$$

where the superscript $\cdot$ represents the interior solution ($\cdot=I$) or the exterior solution ($\cdot=E$), depending on the location of $\mathbf{x}$, i.e.,

$$\mathbb{S}^{\cdot,\infty}_{ijk\ell}(\mathbf{x}) = \begin{cases} \mathbb{S}^{I,\infty}_{ijk\ell}(\mathbf{x}), & \forall\, \mathbf{x} \in \Omega_I \\ \mathbb{S}^{E,\infty}_{ijk\ell}(\mathbf{x}), & \forall\, \mathbf{x} \in \mathbb{R}^3/\Omega_I \end{cases} \tag{18}$$

For spherical inclusions in an infinite elastic medium, the Eshelby tensors have the elementary form (e.g., Refs. [13,14]

1. Interior solution

$$\mathbb{S}_{ijmn}^{I,\infty}(\mathbf{x}) = \frac{(5\nu-1)}{15(1-\nu)}\delta_{ij}\delta_{mn} + \frac{(4-5\nu)}{15(1-\nu)}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}), \quad \mathbf{x}\in\Omega_I \tag{19}$$

2. Exterior solution

$$\begin{aligned}\mathbb{S}_{ijmn}^{E,\infty}(\mathbf{x}) = \frac{\rho^3}{30(1-\nu)}&[(3\rho^2+10\nu-5)\delta_{ij}\delta_{mn}+(3\rho^2-10\nu+5)\\ &\times(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+15(1-\rho^2)\delta_{ij}\bar{x}_m\bar{x}_n+15(1-2\nu\\ &-\rho^2)\delta_{mn}\bar{x}_i\bar{x}_j+15(\nu-\rho^2)(\delta_{im}\bar{x}_j\bar{x}_n+\delta_{jm}\bar{x}_i\bar{x}_n+\delta_{in}\bar{x}_j\bar{x}_m\\ &+\delta_{jn}\bar{x}_i\bar{x}_m)+15(7\rho^2-5)\bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n], \quad \mathbf{x}\in\mathbb{R}^3/\Omega_I\end{aligned} \tag{20}$$

where $\rho := a/|\mathbf{x}|$ and $|\mathbf{x}|=\sqrt{x_ix_i},\ i=1,2,3$.

Inspired by Eq. (17), we postulate the following form of the two considered BVPs

$$\epsilon_{ij}^d(\mathbf{x}) = \mathbb{S}_{ijk\ell}^{\cdot,\star}(\mathbf{x})\epsilon_{k\ell}^*, \quad \forall\,\mathbf{x}\in\Omega \tag{21}$$

where $\mathbb{S}_{ijk\ell}^{\cdot,\star}(\mathbf{x})$ is an unknown fourth-order tensor (the finite Eshelby tensor), we are seeking to obtain. As before the superscript $\cdot$ represents the interior solution ($\cdot=I$) or the exterior solution ($\cdot=E$). The superscript $\star$ stands for the Dirichlet–Eshelby tensor ($\star=D$) or the Neumann–Eshelby tensor ($\star=N$). As a special case we expect to obtain the original infinite Eshelby tensor ($\star=\infty$).

In principle, one may be able to use spherical harmonics to represent the general solution of Eqs. (14) and (15) based on symmetry, but the solution procedure is very much involved. In fact, no explicit solution has been found as shown in a particular case worked out by Luo and Weng [10].

3 The Radial Isotropic Tensor

To solve Eqs. (14) and (15), we first introduce a novel concept of the *radial transversely isotropic tensor*, or in short, the *radial isotropic tensor*.

It may be observed from the expressions of $\mathbb{S}_{ijmn}^{I,\infty}(\mathbf{x})$ and $\mathbb{S}_{ijmn}^{E,\infty}(\mathbf{x})$ above that there appear six independent tensorial bases, which can be arranged in an array as follows

$$\boldsymbol{\Theta}_{ijmn}(\bar{\mathbf{x}}) := \begin{bmatrix} \delta_{ij}\delta_{mn} \\ \delta_{im}\delta_{jn}+\delta_{in}\delta_{jm} \\ \delta_{ij}\bar{x}_m\bar{x}_n \\ \delta_{mn}\bar{x}_i\bar{x}_j \\ \delta_{im}\bar{x}_j\bar{x}_n+\delta_{in}\bar{x}_j\bar{x}_m+\delta_{jm}\bar{x}_i\bar{x}_n+\delta_{jn}\bar{x}_i\bar{x}_m \\ \bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n \end{bmatrix} \tag{22}$$

We term this array as the circumference basis of the Eshelby tensor. By using $\boldsymbol{\Theta}_{ijmn}(\bar{\mathbf{x}})$, both the original interior and exterior Eshelby tensor can be recast into a canonical form, the dot product of two arrays, i.e.,

$$\begin{aligned}\mathbb{S}_{ijmn}^{\cdot,\infty}(\mathbf{x}) &= S_1^{\cdot,\infty}(t)\delta_{ij}\delta_{mn}+S_2^{\cdot,\infty}(t)(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+S_3^{\cdot,\infty}(t)\delta_{ij}\bar{x}_m\bar{x}_n\\ &\quad+S_4^{\cdot,\infty}(t)\delta_{mn}\bar{x}_i\bar{x}_j+S_5^{\cdot,\infty}(t)(\delta_{im}\bar{x}_j\bar{x}_n+\delta_{in}\bar{x}_j\bar{x}_m+\delta_{jm}\bar{x}_i\bar{x}_n\\ &\quad+\delta_{jn}\bar{x}_i\bar{x}_m)+S_6^{\cdot,\infty}(t)\bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n\\ &=\boldsymbol{\Theta}_{ijmn}^T(\bar{\mathbf{x}})\mathbf{S}^{\cdot,\infty}(t).\end{aligned} \tag{23}$$

The arrays, $\mathbf{S}^{I,\infty}(t)$ and $\mathbf{S}^{E,\infty}(t)$, are termed the radial basis of the infinite Eshelby tensor. In accordance to Eqs. (19) and (20) they are given as

$$\mathbf{S}^{I,\infty}(t) = \frac{1}{15(1-\nu)}\begin{bmatrix}5\nu-1\\4-5\nu\\0\\0\\0\\0\end{bmatrix},$$

$$\mathbf{S}^{E,\infty}(t) = \frac{\rho_0^3/t^3}{30(1-\nu)}\begin{bmatrix}3\rho_0^2/t^2+10\nu-5\\3\rho_0^2/t^2-10\nu+5\\15(1-\rho_0^2/t^2)\\15(1-2\nu-\rho_0^2/t^2)\\15(\nu-\rho_0^2/t^2)\\15(7\rho_0^2/t^2-5)\end{bmatrix} \tag{24}$$

where $t=|\mathbf{x}|/A=\rho_0/\rho$, with $\rho=a/|\mathbf{x}|$, and $\rho_0=a/A$.

The above heuristic discussion reveals an important fact, that the Eshelby tensor is a so-called "radial isotropic tensor," which is a generalization of an isotropic tensor. Here, we define the *radial isotropic tensor* as a transversely isotropic tensor along a given radial direction, i.e., a tensor whose properties in all directions perpendicular to the radial direction, $\bar{\mathbf{x}}$, are the same. In general, the radial isotropic tensor, depending on $\mathbf{x}=tA\bar{\mathbf{x}}$, can be expressed in the following canonical form

$$\begin{aligned}\mathbb{S}_{ijmn}(\mathbf{x}) &= S_1(t)\delta_{ij}\delta_{mn}+S_2(t)(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+S_3(t)\delta_{ij}\bar{x}_m\bar{x}_n\\ &\quad+S_4(t)\delta_{mn}\bar{x}_i\bar{x}_j+S_5(t)(\delta_{im}\bar{x}_j\bar{x}_n+\delta_{in}\bar{x}_j\bar{x}_m+\delta_{jm}\bar{x}_i\bar{x}_n\\ &\quad+\delta_{jn}\bar{x}_i\bar{x}_m)+S_6(t)\bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n\\ &=\boldsymbol{\Theta}_{ijmn}^T(\bar{\mathbf{x}})\mathbf{S}(t)\end{aligned} \tag{25}$$

This canonical form decomposes $\mathbb{S}_{ijmn}$ into the circumference basis $\boldsymbol{\Theta}_{ijmn}$, which is only a function of the direction vector $\bar{\mathbf{x}}$, and into the radial basis $\mathbf{S}$, which is only a function of the dimensionless radial distance t.

It is well known that a transversely isotropic tensor has the similar symmetric properties (e.g., Ref. [15]). Using the definition

$$a_{ij} := \delta_{ij} - r_ir_j \tag{26}$$

$$b_{ij} := r_ir_j \tag{27}$$

which are the idempotent parts of a second-order unit tensor,

$$\delta_{ij} = a_{ij} + b_{ij} \tag{28}$$

one can show that the following six bases [15]

$$\mathbb{E}_{ijmn}^1 = \tfrac{1}{2}a_{ij}a_{mn} \tag{29}$$

$$\mathbb{E}_{ijmn}^2 = b_{ij}b_{mn} = r_ir_jr_mr_n \tag{30}$$

$$\mathbb{E}_{ijmn}^3 = \tfrac{1}{2}(a_{im}a_{jn}+a_{jm}a_{in}-a_{ij}a_{mn}) \tag{31}$$

$$\mathbb{E}_{ijmn}^4 = \tfrac{1}{2}(a_{im}b_{jn}+a_{jn}b_{jm}+a_{jn}b_{im}+a_{jm}b_{in}) \tag{32}$$

$$\mathbb{E}_{ijmn}^5 = a_{ij}b_{mn} \tag{33}$$

$$\mathbb{E}_{ijmn}^6 = b_{ij}a_{mn} \tag{34}$$

form a finite non-Abelian group. Furthermore

$$\mathbf{E}^p:\mathbf{E}^q = \mathbf{E}^p \quad \text{if } p=q, \quad \mathbf{E}^p:\mathbf{E}^q = 0 \quad \text{if } p\neq q, \quad p,q=1,2,3,4 \tag{35}$$

where $\mathbf{E}^p=\mathbb{E}_{ijmn}^p\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_m\otimes\mathbf{e}_n$. Subsequently, for $p=1,2,3,4,5,6$, one can find the "less congenial multiplication table" shown in Ref. [15].

Nevertheless, to the best of the authors' knowledge, we are the first to show that the circumference basis $\boldsymbol{\Theta}_{ijmn}$ of a spherical inclusion in a finite domain is a transversely isotropic tensor.[2] Instead of using the partially idempotent canonical form to represent the circumference basis $\mathbf{\Theta}_{ijmn}$ of a radial transversely isotropic tensor, we use an equivalent but different description introduced in Eq. (22).

Based on the symmetry of the problem, we now postulate that the Eshelby tensor for a finite RVE, $\mathbb{S}^{\cdot,\star}_{ijmn}$, should also be a radial isotropic tensor. It therefore admits the following multiplicative decomposition

$$\mathbb{S}^{\cdot,\star}_{ijmn}(\mathbf{x}) = \mathbf{\Theta}^T_{ijmn}(\overline{\mathbf{x}})\mathbf{S}^{\cdot,\star}(t) \tag{36}$$

with the superscripts, $\cdot = I$ or E and $\star = D$ or N. Here $\mathbf{\Theta}_{ijmn}(\overline{\mathbf{x}})$ is the circumference basis according to Eq. (22) and $\mathbf{S}^{\cdot,\star}(t)$ is the radial basis given as

$$\mathbf{S}^{\cdot,\star}(t) = [S^{\cdot,\star}_1(t), S^{\cdot,\star}_2(t), S^{\cdot,\star}_3(t), S^{\cdot,\star}_4(t), S^{\cdot,\star}_5(t), S^{\cdot,\star}_6(t)]^T \tag{37}$$

The scalar entries $S^{\cdot,\star}_J(t)$, $J=1,2,3,4,5,6$, are unknown functions of the nondimensional radial variable $t=|\mathbf{x}|/A$, which are to be determined.

The postulate above is motivated by the following two considerations. Due to the concentric and spherical symmetry of inclusion and RVE the tensorial basis of the finite Eshelby tensor can only depend on the radial direction vector $\overline{x}_i$ (and the second-order identity δ_{ij}). Therefore its tensorial basis, can only consist of combinations of zeroth-, second-, and fourth-order homogeneous functions of $\overline{\mathbf{x}}$. Furthermore due to the symmetry of the strain tensor the finite Eshelby tensor must have minor symmetries. Its tensorial basis can therefore only admit the six tensorial bases listed in $\mathbf{\Theta}_{ijmn}(\overline{\mathbf{x}})$. We note that one should expect more than six bases for problems described by more that one vector, such as ellipsoidal inclusions or non-concentrically placed inclusions within the RVE. Such problems may also be solvable with a similar procedure to ours. Due to the postulate the search for the finite Eshelby tensors reduces to the search for their radial basis $\mathbf{S}^{\cdot,\star}(t)$. We will see in the subsequent section, that the two solutions we obtain satisfy the governing equations exactly, thereby justifying postulate Eq. (36).

In analogy to Eq. (21), we can express the disturbance displacement field as

$$u^d_i(\mathbf{x}) = \begin{cases} \mathbb{U}^{I,\star}_{imn}(\mathbf{x})\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega_I \\ \mathbb{U}^{E,\star}_{imn}(\mathbf{x})\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega_E \end{cases} \tag{38}$$

where $\mathbb{U}^{I,\star}_{imn}(\mathbf{x})$ is a third-order radial isotropic tensor, whose relation to $\mathbb{S}^{I,\star}_{ijmn}(\mathbf{x})$ is discussed next. The disturbance strain is linked to the displacement field by the relation

$$\begin{aligned}\epsilon^d_{ij}(\mathbf{x}) &= \tfrac{1}{2}[u^d_{i,j}(\mathbf{x}) + u^d_{j,i}(\mathbf{x})] = \tfrac{1}{2}[\mathbb{U}^{\cdot,\star}_{imn,j}(\mathbf{x}) + \mathbb{U}^{\cdot,\star}_{jmn,i}(\mathbf{x})]\epsilon^*_{mn} \\ &= \mathbb{S}^{\cdot,\star}_{ijmn}(\mathbf{x})\epsilon^*_{mn}\end{aligned} \tag{39}$$

It can be shown that $\mathbb{U}^{\cdot,\star}_{imn}(\mathbf{x})$ can only admit the following multiplicative decomposition, so that the related Eshelby tensors $\mathbb{S}^{I,\star}_{ijmn}(\mathbf{x})$ are radial isotropic tensors

$$\mathbb{U}^{I,\star}_{imn}(\mathbf{x}) = \mathbf{\Xi}^T_{imn}(\overline{\mathbf{x}})\mathbf{U}^{I,\star}(t), \quad \forall\, \mathbf{x} \in \Omega_I \tag{40}$$

$$\mathbb{U}^{E,\star}_{imn}(\mathbf{x}) = \mathbf{\Xi}^T_{imn}(\overline{\mathbf{x}})\mathbf{U}^{E,\star}(t), \quad \forall\, \mathbf{x} \in \Omega_E \tag{41}$$

with the appearing arrays defined as

$$\mathbf{U}^{I,\star}(t) = \begin{bmatrix} U^{I,\star}_1(t) \\ U^{I,\star}_2(t) \\ U^{I,\star}_3(t) \end{bmatrix}, \quad \mathbf{U}^{E,\star}(t) = \begin{bmatrix} U^{E,\star}_1(t) \\ U^{E,\star}_2(t) \\ U^{E,\star}_3(t) \end{bmatrix}$$

and

$$\mathbf{\Xi}_{imn}(\overline{\mathbf{x}}) = \begin{bmatrix} \overline{x}_i\delta_{mn} \\ \overline{x}_m\delta_{in} + \overline{x}_n\delta_{im} \\ \overline{x}_i\overline{x}_m\overline{x}_n \end{bmatrix} \tag{42}$$

Here $\mathbf{U}^{I,\star}(t)$ and $\mathbf{U}^{E,\star}(t)$ are the radial basis arrays of the displacement field. $\mathbf{\Xi}_{imn}(\overline{\mathbf{x}})$ is the circumference basis array of the displacement field, whose third-order tensorial entries can only be first- or third-order homogeneous function of $\overline{\mathbf{x}}$. Hence the disturbance displacement field has the following canonical form

$$u^d_i(\mathbf{x}) = u^d_i(\overline{\mathbf{x}},t) = \begin{cases} \epsilon^*_{mn}\mathbf{\Xi}^T_{imn}(\overline{\mathbf{x}})\mathbf{U}^{I,\star}(t), & \forall \mathbf{x} \in \Omega_I \\ \epsilon^*_{mn}\mathbf{\Xi}^T_{imn}(\overline{\mathbf{x}})\mathbf{U}^{E,\star}(t), & \forall \mathbf{x} \in \Omega_E \end{cases} \tag{43}$$

Furthermore, the kinematic relation (39) yields the following differential mapping, which uniquely determines the relationship between the radial basis array of the strain field and the radial basis array of the displacement field

$$\mathbf{S}^{\cdot,\star}(t) = \mathcal{D}(t)\mathbf{U}^{\cdot,\star}(t) \tag{44}$$

where $\mathcal{D}(t)$ is a differential operator that is defined in matrix form

$$\mathcal{D}(t) = \frac{1}{A}\begin{bmatrix} \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{t} \\ -\frac{1}{t} + \frac{d}{dt} & 0 & 0 \\ 0 & -\frac{1}{2t} + \frac{1}{2}\frac{d}{dt} & \frac{1}{2t} \\ 0 & 0 & -\frac{3}{t} + \frac{d}{dt} \end{bmatrix}^{6\times 3} \tag{45}$$

Likewise, if $\mathbf{S}$ is given $\mathbf{U}$ can be determined from

$$\mathbf{U}^{\cdot,\star}(t) = \mathfrak{I}(t)\mathbf{S}^{\cdot,\star}(t) \tag{46}$$

where $\mathfrak{I}(t)$ is the integration operator

$$\mathfrak{I}(t) = tA\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{3\times 6} \tag{47}$$

We note that the displacements are only uniquely determinable up to the rigid body motion, which is set to zero here.

4 Eshelby Tensors for Finite Domains

For simplicity, in the rest of the paper, we term the Eshelby tensor for a finite domain as the finite Eshelby tensor.

4.1 The Dirichlet–Eshelby Tensor. We first consider the Dirichlet BVP Eq. (9) in which case $\star = D$. Substituting Eq. (21) into Eq. (16), one obtains a tensorial integral equation for the unknown finite Eshelby tensor

$$\begin{aligned}\mathbb{S}^{\cdot,D}_{ijmn}(\mathbf{x}) = \mathbb{S}^{\cdot,\infty}_{ijmn}(\mathbf{x}) + \frac{1}{2}\int_{\partial\Omega}&[G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})] \\ &\times n_\ell(\mathbf{y})\mathbb{C}_{k\ell pq}\mathbb{S}^{E,D}_{pqmn}(\mathbf{y})d\Omega_y\end{aligned} \tag{48}$$

This integral equation has two different forms, depending on whether $\mathbf{x}$ is inside or outside the inclusion

$$\begin{aligned}\mathbb{S}^{I,D}_{ijmn}(\mathbf{x}) = \mathbb{S}^{I,\infty}_{ijmn}(\mathbf{x}) + \frac{1}{2}\int_{\partial\Omega}&[G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})] \\ &\times n_\ell(\mathbf{y})\mathbb{C}_{k\ell pq}\mathbb{S}^{E,D}_{pqmn}(\mathbf{y})d\Omega_y \quad \forall\, \mathbf{x} \in \Omega_I\end{aligned} \tag{49}$$

[2]We first derived the result in a 2004 manuscript that was submitted to Proceedings of Royal Society of London.

$$\mathbb{S}^{E,D}_{ijmn}(\mathbf{x}) = \mathbb{S}^{E,\infty}_{ijmn}(\mathbf{x}) + \frac{1}{2}\int_{\partial\Omega}[G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})] \times n_\ell(\mathbf{y})\mathbb{C}_{k\ell pq}\mathbb{S}^{E,D}_{pqmn}(\mathbf{y})d\Omega_y \quad \forall\, \mathbf{x} \in \Omega_E \tag{50}$$

Since $\mathbf{y}$ lies on the boundary, $\mathbf{y} \in \partial\Omega$, and since $\mathbf{y}=tA\bar{\mathbf{y}}$ with $t=1$, $\bar{\mathbf{y}}=\mathbf{n}$, we can use Eq. (36) to write $\mathbb{S}^{E,D}_{ijmn}(\mathbf{y})=\mathbf{\Theta}^T_{ijmn}(\mathbf{n})\mathbf{S}^{E,D}(1)$. The postulate, that the circumference basis of the Eshelby tensors for a finite spherical RVE is the same as for the Eshelby tensors in an infinite domain, is true only if the circumference basis is invariant under the boundary integral in Eqs. (49) and (50). This means that the Dirichlet boundary integral, which we denote by $\mathbb{S}^{B,D}_{ijmn}$, can be expressed in terms of the canonical form

$$\begin{aligned}\mathbb{S}^{B,D}_{ijmn}(\mathbf{x}) &= \frac{1}{2}\int_{\partial\Omega}\mathbb{C}_{k\ell st}[G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) \\ &\quad + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})]n_\ell(\mathbf{y})\mathbf{\Theta}^T_{stmn}(\mathbf{n})\mathbf{S}^{E,D}(1)dS_y \\ &= \int_{\partial\Omega}\mathbb{G}_{ijmn}(\mathbf{x},\mathbf{y})dS_y = \mathbf{\Theta}^T_{ijmn}(\bar{\mathbf{x}})\mathbf{S}^{B,D}(t)\end{aligned} \tag{51}$$

Here $\mathbb{G}_{ijmn}$ is the integrand of the boundary integral which follows as

$$\begin{aligned}\mathbb{G}_{ijmn}(\mathbf{x},\mathbf{y}) &= \frac{1}{2}\mathbb{C}_{k\ell st}[G^{\infty}_{ik,j}(\mathbf{x}-\mathbf{y}) + G^{\infty}_{jk,i}(\mathbf{x}-\mathbf{y})] \\ &\quad \times n_\ell(\mathbf{y})\mathbf{\Theta}^T_{stmn}(\mathbf{n})\mathbf{S}^{E,D}(1) \\ &= \frac{-1}{16\pi(1-\nu)\mu r^2}[T_1\bar{r}_k n_k\delta_{ij}\delta_{mn} + T_1(2\nu-1)(\bar{r}_i n_j \\ &\quad + \bar{r}_j n_i)\delta_{mn} + T_2(\bar{r}_m n_n + \bar{r}_n n_m)\delta_{ij} + T_2(2\nu-1)(\delta_{im}\bar{r}_j n_n \\ &\quad + \delta_{in}\bar{r}_j n_m + \delta_{jm}\bar{r}_i n_n + \delta_{jn}\bar{r}_i n_m) - 3T_1\bar{r}_i\bar{r}_j\bar{r}_k n_k\delta_{mn} \\ &\quad + T_3\bar{r}_k n_k n_m n_n\delta_{ij} - 3T_2(\bar{r}_i\bar{r}_j\bar{r}_m n_n + \bar{r}_i\bar{r}_j\bar{r}_n n_m) + T_3(2\nu \\ &\quad - 1)(\bar{r}_i n_j n_m n_n + \bar{r}_j n_i n_m n_n) - 3T_3\bar{r}_i\bar{r}_j\bar{r}_k n_k n_m n_n]\end{aligned} \tag{52}$$

where $\bar{r}_i=(y_i-x_i)/r$, $r=|\mathbf{y}-\mathbf{x}|$ and $n_i(\mathbf{y})=y_i/|\mathbf{y}|$ for $\mathbf{y} \in \partial\Omega$. The coefficient vector

$$\mathbf{T} = [T_1, T_2, T_3]^T \tag{53}$$

is a stress projection vector (see Sec. 4.3), which follows as $\mathbf{T}=\mathbf{K}_1\mathbf{S}^{E,D}(1)$ where

$$\mathbf{K}_1 = \mu\begin{bmatrix} \frac{2+2\nu}{1-2\nu} & \frac{4\nu}{1-2\nu} & 0 & \frac{2(1-\nu)}{1-2\nu} & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & \frac{2+2\nu}{1-2\nu} & 0 & \frac{4}{1-2\nu} & \frac{2(1-\nu)}{1-2\nu}\end{bmatrix} \tag{54}$$

With the aid of the following integrals (see the Appendix)

$$\text{(I)} \quad \int_{\partial\Omega}\frac{1}{r^2}\bar{r}_k n_k\, dS_y = 4\pi \tag{55}$$

$$\text{(I)} \quad \int_{\partial\Omega}\frac{1}{r^2}\bar{r}_i n_j\, dS_y = \frac{4\pi}{3}\delta_{ij} \tag{56}$$

$$\text{(III)} \quad \int_{\partial\Omega}\frac{1}{r^2}\bar{r}_i\bar{r}_j\bar{r}_k n_k\, dS_y = \frac{4\pi}{3}\delta_{ij} \tag{57}$$

$$\text{(IV)} \quad \int_{\partial\Omega}\frac{1}{r^2}\bar{r}_k n_k n_m n_n\, dS_y = \frac{4\pi}{15}(5-3t^2)\delta_{mn} + \frac{12\pi}{5}t^2\bar{x}_m\bar{x}_n \tag{58}$$

$$\text{(V)} \quad \int_{\partial\Omega}\frac{1}{r^2}\bar{r}_i\bar{r}_j(\bar{r}_m n_n + \bar{r}_n n_m)dS_y = \frac{8\pi}{15}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) \tag{59}$$

$$\begin{aligned}\text{(VI)} \quad &\int_{\partial\Omega}\frac{1}{r^2}(\bar{r}_i n_j + \bar{r}_j n_i)n_m n_n\, dS_y \\ &= \frac{\pi}{105}[(56-24t^2)(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) + 120t^2\delta_{ij}\bar{x}_m\bar{x}_n \\ &\quad - 48t^2\bar{x}_i\bar{x}_j\delta_{mn} + 36t^2(\delta_{im}\bar{x}_j\bar{x}_n + \delta_{in}\bar{x}_j\bar{x}_m \\ &\quad + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m)]\end{aligned} \tag{60}$$

$$\begin{aligned}\text{(VII)} \quad &\int_{\partial\Omega}\frac{1}{r^2}\bar{r}_i\bar{r}_j\bar{r}_k n_k n_m n_n\, dS_y \\ &= \frac{\pi}{105}[(28-20t^2)(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) + 100t^2\delta_{ij}\bar{x}_m\bar{x}_n \\ &\quad + 16t^2\bar{x}_i\bar{x}_j\delta_{mn} - 12t^2(\delta_{im}\bar{x}_j\bar{x}_n + \delta_{in}\bar{x}_j\bar{x}_m \\ &\quad + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m)]\end{aligned} \tag{61}$$

we obtain the boundary contribution by explicit integration as

$$\mathbb{S}^{B,D}_{ijmn}(\mathbf{x}) = \mathbf{\Theta}^T_{ijmn}(\bar{\mathbf{x}})\mathbf{S}^{B,D}(t) \tag{62}$$

with

$$\mathbf{S}^{B,D}(t) = \mathbf{K}_2(t)\mathbf{K}_1\mathbf{S}^{E,D}(1)$$

and

$$\mathbf{K}_2(t) = \frac{-1}{420(1-\nu)} \times \begin{bmatrix} 70(2\nu-1) & 28 & 4\nu(7-3t^2) \\ 0 & 28(5\nu-4) & 7(4\nu-5)+3t^2(7-4\nu) \\ 0 & 0 & 6t^2(10\nu-7) \\ 0 & 0 & -24\nu t^2 \\ 0 & 0 & 18\nu t^2 \\ 0 & 0 & 0 \end{bmatrix} \tag{63}$$

The integral Eqs. (49) and (50) can then be reduced to a pair of algebraic evolution equations,

$$\mathbf{\Theta}^T_{ijmn}(\bar{\mathbf{x}})\mathbf{S}^{I,D}(t) = \mathbf{\Theta}^T_{ijmn}(\bar{\mathbf{x}})[\mathbf{S}^{I,\infty}(t) + \mathbf{K}_3(t)\mathbf{S}^{E,D}(1)] \tag{64}$$

$$\mathbf{\Theta}^T_{ijmn}(\bar{\mathbf{x}})\mathbf{S}^{E,D}(t) = \mathbf{\Theta}^T_{ijmn}(\bar{\mathbf{x}})[\mathbf{S}^{E,\infty}(t) + \mathbf{K}_3(t)\mathbf{S}^{E,D}(1)] \tag{65}$$

where $\mathbf{K}_3(t) \coloneqq \mathbf{K}_2(t)\mathbf{K}_1$. The matrix $\mathbf{K}_2(t)$ maps the effect of the boundary traction onto the domain of the RVE.

Eliminating the circumference basis from Eqs. (64) and (65), we derive the following parametric algebraic equations

$$\mathbf{S}^{I,D}(t) = \mathbf{S}^{I,\infty}(t) + \mathbf{K}_3(t)\mathbf{S}^{E,D}(1), \quad 0 \leqslant t < \rho_0 \tag{66}$$

$$\mathbf{S}^{E,D}(t) = \mathbf{S}^{E,\infty}(t) + \mathbf{K}_3(t)\mathbf{S}^{E,D}(1), \quad \rho_0 \leqslant t < 1 \tag{67}$$

Let us assume that $\mathbf{S}^{E,D}(t)$ continuously depends on t so that

$$\lim_{t\to 1}\mathbf{S}^{E,D}(t) \to \mathbf{S}^{E,D}(1) \tag{68}$$

Now let $t \to 1$ in Eq. (67), so that we can obtain $\mathbf{S}^{E,D}(1)$ by solving Eq. (67), i.e.,

$$\mathbf{S}^{E,D}(1) = [\mathbf{I} - \mathbf{K}_3(1)]^{-1}\mathbf{S}^{E,\infty}(1) = \rho_0^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{7-28\nu+20\nu^2-7(1-2\nu)\rho_0^2}{2(1-\nu)(7-10\nu)} \\ \dfrac{10\nu-7\rho_0^2}{2\ (7-10\nu)} \\ \dfrac{7\rho_0^2-5}{2(1-\nu)} \end{bmatrix} \tag{69}$$

The boundary contribution $\mathbf{S}^{B,D}(t)=\mathbf{K}_3(t)\mathbf{S}^{E,D}(1)$ then becomes

$$\mathbf{S}^{B,D}(t) = -\frac{\rho_0^3}{15(1-\nu)}\begin{bmatrix} 5\nu-1 \\ 4-5\nu \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{\rho_0^3(1-\rho_0^2)}{20(1-\nu)(7-10\nu)}\begin{bmatrix} 2(7-10\nu t^2) \\ 7(5t^2-3)-20\nu t^2 \\ -10t^2(7-10\nu) \\ -40\nu t^2 \\ 30\nu t^2 \\ 0 \end{bmatrix} \tag{70}$$

With this the considered Dirichlet–Eshelby problem of a spherical RVE is fully solved. The radial basis arrays of the Dirichlet–Eshelby tensors is given by

$$\mathbf{S}^{I,D}(t) = \mathbf{S}^{I,\infty}(t) + \mathbf{S}^{B,D}(t), \quad 0 \leq t < \rho_0 \tag{71}$$

$$\mathbf{S}^{E,D}(t) = \mathbf{S}^{E,\infty}(t) + \mathbf{S}^{B,D}(t), \quad \rho_0 \leq t \leq 1 \tag{72}$$

and from Eq. (36) we finally obtain the Dirichlet–Eshelby tensors. The interior solution is

$$\begin{aligned} \mathbb{S}_{ijmn}^{I,D}(\mathbf{x}) = \frac{1}{1-\nu}\Bigg\{ & \left[\frac{5\nu-1}{15}(1-\rho_0^3) + \frac{7-10\nu t^2}{10(7-10\nu)}\rho_0^3(1-\rho_0^2)\right]\delta_{ij}\delta_{mn} \\ & + \left[\frac{4-5\nu}{15}(1-\rho_0^3) + \frac{7(5t^2-3)-20\nu t^2}{20(7-10\nu)}\rho_0^3(1-\rho_0^2)\right] \\ & \times(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) - \frac{t^2}{2}\rho_0^3(1-\rho_0^2)\delta_{ij}\bar{x}_m\bar{x}_n - \frac{2\nu t^2}{7-10\nu}\rho^3(1 \\ & - \rho_0^2)\delta_{mn}\bar{x}_i\bar{x}_j + \frac{3\nu t^2}{2(7-10\nu)}\rho_0^3(1-\rho_0^2)(\delta_{im}\bar{x}_j\bar{x}_n + \delta_{in}\bar{x}_j\bar{x}_m \\ & + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m)\Bigg\} \end{aligned} \tag{73}$$

and the exterior solution is

$$\begin{aligned} \mathbb{S}_{ijmn}^{E,D}(\mathbf{x}) = \frac{\rho_0^3}{1-\nu}\Bigg\{ & \left[\frac{3\rho_0^2/t^2+10\nu-5}{30t^3} - \frac{5\nu-1}{15} + \frac{7-10\nu t^2}{10(7-10\nu)}(1-\rho_0^2)\right]\delta_{ij}\delta_{mn} \\ & + \left[\frac{3\rho_0^2/t^2-10\nu+5}{30t^3} - \frac{4-5\nu}{15} + \frac{7(5t^2-3)-20\nu t^2}{20(7-10\nu)}(1-\rho_0^2)\right](\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) \\ & - \left[\frac{\rho_0^2/t^2-1}{2t^3} + \frac{t^2}{2}(1-\rho_0^2)\right]\delta_{ij}\bar{x}_m\bar{x}_n - \left[\frac{\rho_0^2/t^2+2\nu-1}{2t^3} + \frac{2\nu t^2}{7-10\nu}(1-\rho_0^2)\right]\delta_{mn}\bar{x}_i\bar{x}_j \\ & - \left[\frac{\rho_0^2/t^2-\nu}{2t^3} - \frac{3\nu t^2}{2(7-10\nu)}(1-\rho_0^2)\right](\delta_{im}\bar{x}_j\bar{x}_n + \delta_{in}\bar{x}_j\bar{x}_m + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m) + \frac{7\rho_0^2/t^2-5}{2t^3}\bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n\Bigg\} \end{aligned} \tag{74}$$

We can see that both the interior and the exterior Dirichlet Eshelby tensor are neither constant nor isotropic. The dependency on the position $\mathbf{x}$ is captured by the dependency on $\bar{\mathbf{x}}$ and t. Furthermore, both tensors depend explicitly on the ratio ρ_0 between inclusion and RVE. If we let $\rho_0 \to 0$ we recover the original infinite Eshelby tensors exactly since the boundary contribution then vanishes. To visualize the Dirichlet–Eshelby tensors the profiles of the components of the radial basis arrays $\mathbf{S}^{\cdot,\infty}(t)$, $\mathbf{S}^{\cdot,D}(t)$ and $\mathbf{S}^{B,D}(t)$ are shown in Fig. 2. Here the relative size of the inclusion is chosen as $\rho_0=0.4$, so that the volume fraction becomes $\rho_0^3=0.064$. Poisson's ratio of the matrix phase is picked as $\nu=0.3$. One can clearly observe that the boundary term $\mathbf{S}^{B,D}$, which can be understood as a correction of Eshelby's original result, is substantial. It can also be noted that there is a discontinuity across the interface between the inclusion and the matrix.

The disturbance displacement field $u_i^d(\mathbf{x})$ is now given by Eqs. (38), (40), and (41), i.e.,

$$u_i^d(\mathbf{x}) = \boldsymbol{\Xi}_{imn}^T(\bar{\mathbf{x}})\mathbf{U}^{\cdot,D}(t)\epsilon_{mn}^*, \quad \forall\, \mathbf{x} \in \Omega \tag{75}$$

where the arrays $\boldsymbol{\Xi}_{imn}(\bar{\mathbf{x}})$, $\mathbf{U}^{I,D}(t)$, and $\mathbf{U}^{E,D}(t)$ follow from Eqs. (42) and (46). Applying operator (47) to Eqs. (71), (72), (24), and (70) we easily obtain

$$\mathbf{U}^{I,D}(t) = \mathbf{U}^{I,\infty}(t) + \mathbf{U}^{B,D}(t), \quad 0 \leq t < \rho_0 \tag{76}$$

$$\mathbf{U}^{E,D}(t) = \mathbf{U}^{E,\infty}(t) + \mathbf{U}^{B,D}(t), \quad \rho_0 \leq t \leq 1 \tag{77}$$

with

$$\mathbf{U}^{I,\infty}(t) = \frac{tA}{15(1-\nu)}\begin{bmatrix} 5\nu-1 \\ 4-5\nu \\ 0 \end{bmatrix} \tag{78}$$

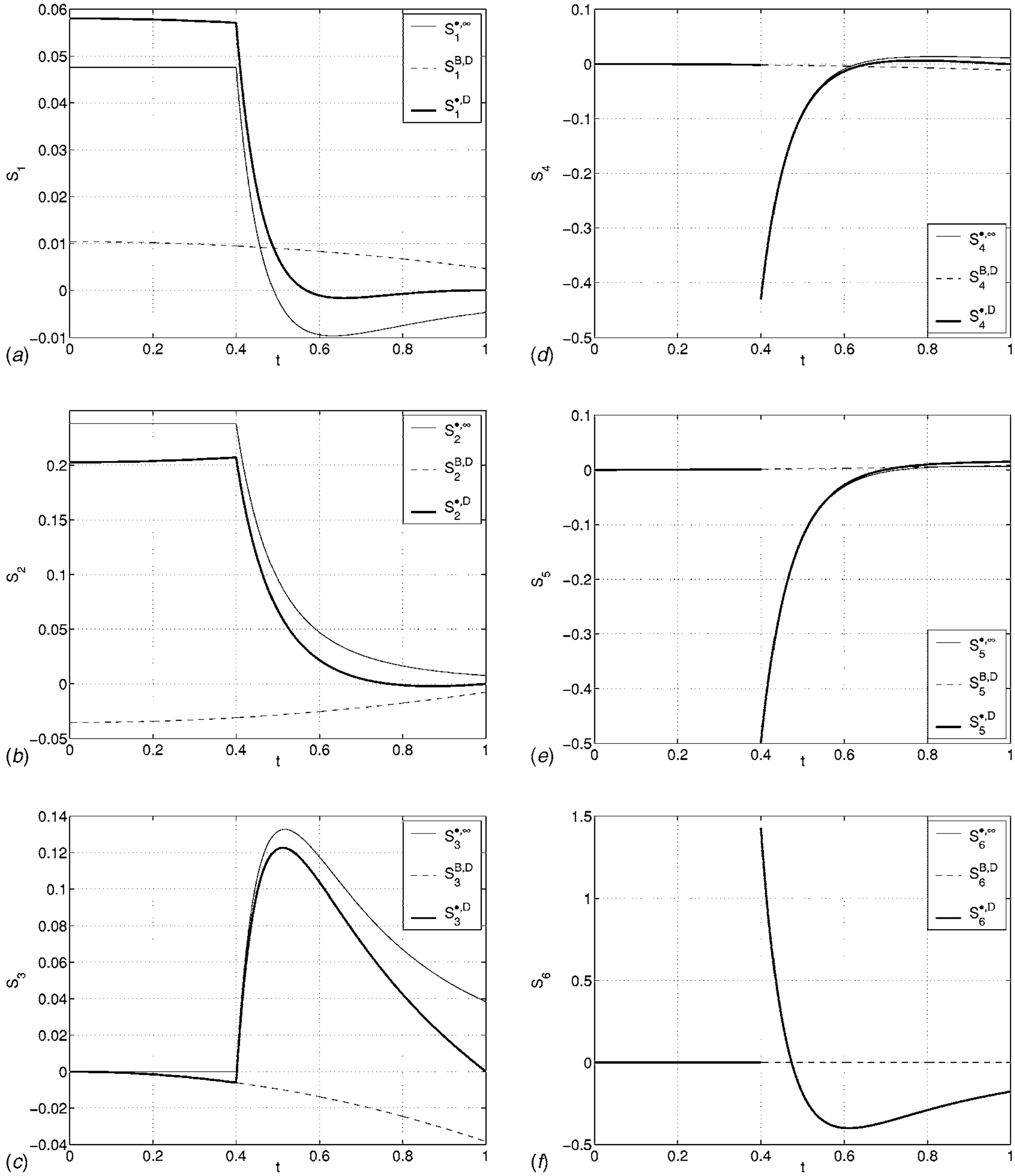

Fig. 2 The components of the radial basis arrays $\mathbf{S}^{\cdot,\infty}$, $\mathbf{S}^{B,D}$, and $\mathbf{S}^{\cdot,D}$

$$\mathbf{U}^{E,\infty}(t) = \frac{\rho_0^3 A}{30t^2(1-\nu)} \begin{bmatrix} 3\rho_0^2/t^2 + 10\nu - 5 \\ 3\rho_0^2/t^2 - 10\nu + 5 \\ 15 - 15\rho_0^2/t^2 \end{bmatrix} \tag{79}$$

and

$$\mathbf{U}^{B,D}(t) = -\frac{\rho_0^3 tA}{15(1-\nu)} \begin{bmatrix} 5\nu - 1 \\ 4 - 5\nu \\ 0 \end{bmatrix} + \frac{\rho_0^3(1-\rho_0^2)tA}{20(1-\nu)(7-10\nu)} \begin{bmatrix} 2(7-10\nu t^2) \\ 7(5t^2-3) - 20\nu t^2 \\ -10t^2(7-10\nu) \end{bmatrix} \tag{80}$$

Here $\mathbf{U}^{I,\infty}(t)$ and $\mathbf{U}^{E,\infty}(t)$ are the radial basis array of Eshelby's classical solution in unbounded space and $\mathbf{U}^{B,D}(t)$ is the radial basis contribution from the Dirichlet boundary of the RVE.

We remark that u_i^d given by the equations above satisfies the Fredholm-type integral equation of the Dirichlet BVP Eq. (14) exactly. Furthermore it is readily verified that when $t=1$

$$\mathbf{U}^{E,D}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow u_i^d(\mathbf{y}) = \epsilon_{mn}^{*} \mathbf{\Xi}_{imn}^{T}(\mathbf{n})\mathbf{U}^{E,D}(1) = 0, \quad \forall\, \mathbf{y} \in \partial\Omega \tag{81}$$

This confirms that the obtained displacement solution does indeed satisfy the Dirichlet boundary condition. The coefficients of the radial bases $\mathbf{U}^{\cdot,\infty}$, $\mathbf{U}^{B,D}$, and $\mathbf{U}^{\cdot,D}$ are displayed in Figs. 3(*a*), 3(*c*), and 3(*e*), where we have chosen $\rho_0=0.4$ and $\nu=0.3$. Again, we observe that the boundary correction is substantial. Further, one can see that the Dirichlet solution satisfies the zero displacement boundary condition exactly.

4.2 The Neumann–Eshelby Tensor. The solution of the Neumann–Eshelby problem (now $\star=N$) is different from the

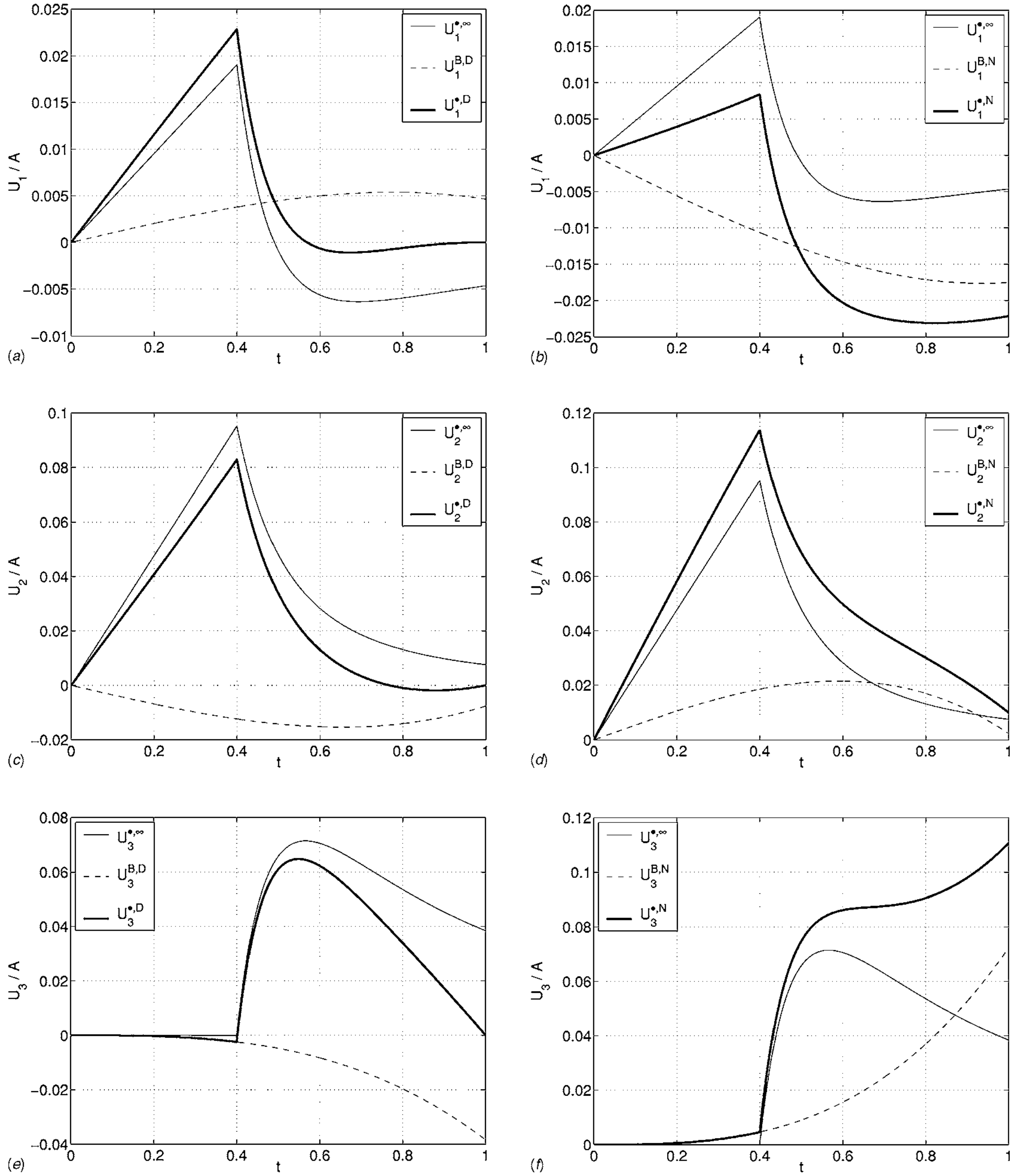

Fig. 3 The components of the radial basis arrays $\mathbf{U}^{\cdot,\infty}$, $\mathbf{U}^{B,D}$, $\mathbf{U}^{\cdot,D}$, $\mathbf{U}^{B,N}$, and $\mathbf{U}^{\cdot,N}$

Dirichlet–Eshelby problem tensor; here the solution is based on the displacement field. For the Neumann BVP Eq. (10), the displacements on the boundary of the RVE ($t=1$) are nonzero and according to Eq. (43) we have

$$u_k^d(\mathbf{y}) = \epsilon_{mn}^* \boldsymbol{\Xi}_{kmn}^T(\mathbf{n})\mathbf{U}^{E,N}(1), \quad \forall\, \mathbf{y} \in \partial\Omega \tag{82}$$

By substituting Eq. (82) into the integral equation corresponding to the Neumann BVP Eq. (15), we obtain an equation for the unknown radial basis, $\mathbf{U}^{\cdot,N}(t)$

$$\begin{aligned}\epsilon_{mn}^* \boldsymbol{\Xi}_{imn}^T(\bar{\mathbf{x}})\mathbf{U}^{\cdot,N}(t) &= -\epsilon_{mn}^* \int_{\Omega_e} \mathbb{C}_{pqmn} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y})\,d\Omega_y \\ &\quad + \epsilon_{mn}^* \int_{\partial\Omega} \mathbb{C}_{pqk\ell} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y})\boldsymbol{\Xi}_{kmn}^T(\mathbf{n})\mathbf{U}^{E,N}(1)\,n_\ell(\mathbf{y})\,dS_y\end{aligned} \tag{83}$$

where $\cdot = I$, or E. Depending on whether $\mathbf{x}$ is inside or outside the inclusion, the domain integral in Eq. (83) has two different forms, which can be expressed in the canonical form

$$-\int_{\Omega_e} \mathbb{C}_{pqmn} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y})\,d\Omega_y = \begin{cases} \boldsymbol{\Xi}_{imn}^T(\bar{\mathbf{x}})\mathbf{U}^{I,\infty}(t), & \forall\, \mathbf{x} \in \Omega_I \\ \boldsymbol{\Xi}_{imn}^T(\bar{\mathbf{x}})\mathbf{U}^{E,\infty}(t), & \forall\, \mathbf{x} \in \Omega_E \end{cases} \tag{84}$$

Here $\mathbf{U}^{I,\infty}(t)$ and $\mathbf{U}^{E,\infty}(t)$ are the radial basis arrays of Eshelby's classical solution for unbounded space (see Eqs. (78) and (79)). In analogy to the Dirichlet case (see Eq. (51)), we stipulate that a similar canonical form holds for the Neumann boundary contribution in Eq. (83)

$$\int_{\partial\Omega} \mathbb{C}_{pqk\ell} G^{\infty}_{pi,q}(\mathbf{x}-\mathbf{y}) \boldsymbol{\Xi}^T_{kmn}(\mathbf{n}) \mathbf{U}^{E,N}(1) n_\ell(\mathbf{y}) dS_y$$

$$= \boldsymbol{\Xi}^T_{imn}(\bar{\mathbf{x}}) \mathbf{U}^{B,N}(t), \quad \forall\, \mathbf{x} \in \Omega \tag{85}$$

where $\mathbf{U}^{B,N}(t)$ denotes the radial basis array arising from the Neumann boundary. Substituting Eqs. (84) and (85) into Eq. (83) and eliminating ϵ^*_{mn} and the circumference basis $\Xi^T_{imn}(\bar{\mathbf{x}})$, one may reduce Eq. (83) into a pair of parametric, algebraic equations for the radial basis arrays, $\mathbf{U}^{\cdot,N}(t)$, i.e.,

$$\mathbf{U}^{I,N}(t) = \mathbf{U}^{I,\infty}(t) + \mathbf{U}^{B,N}(t), \quad 0 \le t \le \rho_0 \tag{86}$$

$$\mathbf{U}^{E,N}(t) = \mathbf{U}^{E,\infty}(t) + \mathbf{U}^{B,N}(t), \quad \rho_0 \le t \le 1 \tag{87}$$

Here $\mathbf{U}^{I,\infty}(t)$ and $\mathbf{U}^{E,\infty}(t)$ are the radial basis vectors of Eshelby's classical solution of unbounded space (see Eqs. (78) and (79)). The boundary contribution, $\mathbf{U}^{B,N}(t)$ follows directly from Eq. (85) as

$$\boldsymbol{\Xi}^T_{imn}(\bar{\mathbf{x}}) \mathbf{U}^{B,N}(t) = \int_{\partial\Omega} \mathbb{H}_{imn}(\mathbf{x},\mathbf{y}) dS_y \tag{88}$$

with the integrand

$$\begin{aligned}
\mathbb{H}_{imn} &= \mathbb{C}_{pqk\ell} G^{\infty}_{pi,q}(\mathbf{x}-\mathbf{y}) \boldsymbol{\Xi}^T_{kmn}(\mathbf{n}) \mathbf{U}^{E,N}(1) n_\ell(\mathbf{y}) \\
&= \frac{1}{8\pi(1-\nu)r^2} \{ U_1^{E,N}(1)[(1-2\nu)(2n_i n_p \bar{r}_p \delta_{mn} - \bar{r}_i \delta_{mn}) \\
&\quad + 3\bar{r}_i n_p \bar{r}_p n_q \bar{r}_q \delta_{mn}] + U_2^{E,N}(1)[(1-2\nu)(n_m n_p \bar{r}_p \delta_{in} + n_n n_p \bar{r}_p \delta_{im} \\
&\quad + n_i n_m \bar{r}_n + n_i n_n \bar{r}_m - 2n_m n_n \bar{r}_i) + 3n_m \bar{r}_i \bar{r}_n n_p \bar{r}_p + 3n_n \bar{r}_i \bar{r}_m n_p \bar{r}_p] \\
&\quad + U_3^{E,N}(1)[(1-2\nu)(2n_i n_m n_n \bar{r}_p n_p - n_m n_n \bar{r}_i) \\
&\quad + 3n_m n_n \bar{r}_i n_p \bar{r}_p n_q \bar{r}_q]\}
\end{aligned} \tag{89}$$

With the aid of following integrals (see the Appendix)

$$\text{(VIII)} \quad \int_{\partial\Omega} \frac{1}{r^2} \bar{r}_i \, dS_y = 0 \tag{90}$$

$$\text{(IX)} \quad \int_{\partial\Omega} \frac{1}{r^2} n_i n_k \bar{r}_k \, dS_y = \frac{8\pi}{3} t \bar{x}_i \tag{91}$$

$$\text{(X)} \quad \int_{\partial\Omega} \frac{1}{r^2} n_i n_j \bar{r}_k \, dS_y = \frac{4\pi}{15} t(3\bar{x}_i \delta_{jk} + 3\bar{x}_j \delta_{ik} - 2\bar{x}_k \delta_{ij}) \tag{92}$$

$$\text{(XI)} \quad \int_{\partial\Omega} \frac{1}{r^2} n_i \bar{r}_j \bar{r}_k n_p \bar{r}_p \, dS_y = \frac{4\pi}{15} t(4\bar{x}_i \delta_{jk} - \bar{x}_j \delta_{ik} - \bar{x}_k \delta_{ij}) \tag{93}$$

$$\begin{aligned}
\text{(XII)} \quad \int_{\partial\Omega} \frac{1}{r^2} n_i n_j n_k n_p \bar{r}_p \, dS_y &= \frac{\pi}{105}[t(56-48t^2)(\bar{x}_i \delta_{jk} + \bar{x}_j \delta_{ik} \\
&\quad + \bar{x}_k \delta_{ij}) + 240t^3 \bar{x}_i \bar{x}_j \bar{x}_k]
\end{aligned} \tag{94}$$

$$\text{(XIII)} \quad \int_{\partial\Omega} \frac{1}{r^2} \bar{r}_i n_p \bar{r}_p n_q \bar{r}_q \, dS_y = 0 \tag{95}$$

$$\begin{aligned}
\text{(XIV)} \quad \int_{\partial\Omega} \frac{1}{r^2} n_i n_j \bar{r}_k n_p \bar{r}_p n_q \bar{r}_q \, dS_y &= \frac{\pi}{105}[t(84-80t^2)(\bar{x}_i \delta_{jk} + \bar{x}_j \delta_{ik}) \\
&\quad - t(56-32t^2)\bar{x}_k \delta_{ij} + 64t^3 \bar{x}_i \bar{x}_j \bar{x}_k]
\end{aligned} \tag{96}$$

Eq. (88) can be integrated exactly. After some manipulations, the final result can be expressed in a succinct form

$$\mathbf{U}^{B,N}(t) = \mathbf{K}_4(t) \mathbf{U}^{E,N}(1) \tag{97}$$

where

$$\mathbf{K}_4(t) = \frac{t}{1-\nu} \begin{bmatrix} \dfrac{2(1-2\nu)}{3} & \dfrac{2(1-5\nu)}{15} & \dfrac{-2\nu(7-4t^2)}{35} \\ 0 & \dfrac{7-5\nu}{15} & \dfrac{7(5-\nu)+6t^2(4\nu-7)}{105} \\ 0 & 0 & \dfrac{4(7-10\nu)t^2}{35} \end{bmatrix} \tag{98}$$

Equation (97) represents the boundary contribution or "the image contribution" to the disturbance displacement field inside the RVE. Now, the parametric algebraic equations are solely in terms of the displacement radial basis array $\mathbf{U}^{\cdot,N}(t)$

$$\mathbf{U}^{I,N}(t) = \mathbf{U}^{I,\infty}(t) + \mathbf{K}_4(t) \mathbf{U}^{E,N}(1), \quad 0 \le t \le \rho_0 \tag{99}$$

$$\mathbf{U}^{E,N}(t) = \mathbf{U}^{E,\infty}(t) + \mathbf{K}_4(t) \mathbf{U}^{E,F}(1), \quad \rho_0 \le t < 1 \tag{100}$$

We assume that the radial basis array, $\mathbf{U}^{E,N}(t)$, depends continuously on t so that

$$\lim_{t \to 1} \mathbf{U}^{E,N}(t) = \mathbf{U}^{E,N}(1) \tag{101}$$

One can then solve for $\mathbf{U}^{E,N}(1)$ by letting $t=1$ in Eq. (100), i.e.,

$$\mathbf{U}^{E,N}(1) = [\mathbf{I} - \mathbf{K}_4(1)]^{-1} \mathbf{U}^{E,\infty}(1) \tag{102}$$

which gives

$$\mathbf{U}^{E,N}(1) = \frac{\rho_0^3 A}{2(7+5\nu)} \begin{bmatrix} 7(\rho_0^2 - 1) \\ 5\nu + 7\rho_0^2 \\ 35(1-\rho_0^2) \end{bmatrix} \tag{103}$$

Substituting Eq. (103) into Eq. (97), one can evaluate the radial basis array due to the boundary or image contribution

$$\mathbf{U}^{B,N}(t) = \frac{\rho_0^3 t A}{30(1-\nu)} \begin{bmatrix} 2-10\nu \\ 7-5\nu \\ 0 \end{bmatrix} - \frac{\rho_0^3(1-\rho_0^2) t A}{5(1-\nu)(7+5\nu)} \begin{bmatrix} 2(7-10\nu t^2) \\ 7(5t^2-3) - 20\nu t^2 \\ -10t^2(7-10\nu) \end{bmatrix} \tag{104}$$

Note the similarity between the two boundary contributions $\mathbf{U}^{B,N}(t)$ and $\mathbf{U}^{B,D}(t)$ (see Eq. (80)). With the above result we can now find $\mathbf{U}^{I,N}(t)$ and $\mathbf{U}^{E,N}(t)$ from Eqs. (86) and (87).

With the radial basis arrays of the displacement field given, one can apply the differential operator Eq. (45) to obtain the radial basis array of the strain field, i.e.,

$$\mathbf{S}^{I,\infty}(t) = \mathcal{D}(t) \mathbf{U}^{I,\infty}(t), \quad \mathbf{S}^{E,\infty}(t) = \mathcal{D}(t) \mathbf{U}^{E,\infty}(t)$$

and

$$\mathbf{S}^{B,N}(t) = \mathcal{D}(t) \mathbf{U}^{B,N}(t) \tag{105}$$

$\mathbf{S}^{I,\infty}(t)$ and $\mathbf{S}^{E,\infty}(t)$ follow as given in Eq. (24); furthermore by differentiation we find

$$
\mathbf{S}^{B,N}(t) = \frac{\rho_0^3}{30(1-\nu)}\begin{bmatrix} 2-10\nu \\ 7-5\nu \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{\rho_0^3(1-\rho_0^2)}{5(1-\nu)(7+5\nu)}\begin{bmatrix} 2(7-10\nu t^2) \\ 7(5t^2-3)-20\nu t^2 \\ -10t^2(7-10\nu) \\ -40\nu t^2 \\ 30\nu t^2 \\ 0 \end{bmatrix} \tag{106}
$$

In analogy to Eqs. (86) and (87), the radial basis arrays $\mathbf{S}^{\cdot,N}$ of the Neumann–Eshelby tensors now follows from

$$
\mathbf{S}^{I,N}(t) = \mathbf{S}^{I,\infty}(t) + \mathbf{S}^{B,N}(t), \quad 0 \leqslant t \leqslant \rho_0 \tag{107}
$$

$$
\mathbf{S}^{E,N}(t) = \mathbf{S}^{E,\infty}(t) + \mathbf{S}^{B,N}(t), \quad \rho_0 \leqslant t \leqslant 1 \tag{108}
$$

The Neumann–Eshelby tensors for a spherical inclusion embedded in a spherical RVE under the prescribed traction boundary condition can now be obtained from Eq. (22) which yields the following exact and elementary expressions. The interior solution is

$$
\begin{aligned}
\mathbb{S}^{I,N}_{ijmn}(\mathbf{x}) = \frac{1}{1-\nu}\Bigg\{ & \left[\frac{5\nu-1}{15}(1-\rho_0^3) - \frac{2(7-10\nu t^2)}{5(7+5\nu)}\rho_0^3(1-\rho_0^2)\right]\delta_{ij}\delta_{mn} \\
& + \left[\frac{1-\nu}{2} + \frac{5\nu-7}{30}(1-\rho_0^3) - \frac{7(5t^2-3)-20\nu t^2}{5(7+5\nu)}\rho_0^3(1-\rho_0^2)\right](\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) + \frac{2t^2(7-10\nu)}{7+5\nu}\rho_0^3(1-\rho^2)\,\delta_{ij}\bar{x}_m\bar{x}_n \\
& + \frac{8\nu t^2}{7+5\nu}\rho^3(1-\rho_0^2)\,\delta_{mn}\bar{x}_i\bar{x}_j - \frac{6\nu t^2}{7+5\nu}\rho_0^3(1-\rho_0^2)(\delta_{im}\bar{x}_j\bar{x}_n \\
& + \delta_{in}\bar{x}_j\bar{x}_m + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m)\Bigg\}
\end{aligned} \tag{109}
$$

and the exterior solution is

$$
\begin{aligned}
\mathbb{S}^{E,N}_{ijmn}(\mathbf{x}) = \frac{\rho_0^3}{1-\nu}\Bigg\{ & \left[\frac{3\rho^2/t^2+10\nu-5}{30t^3} + \frac{5\nu-1}{15} - \frac{2(7-10\nu t^2)}{5(7+5\nu)}(1-\rho_0^2)\right]\delta_{ij}\delta_{mn} \\
& + \left[\frac{3\rho_0^2/t^2-10\nu+5}{30t^3} + \frac{7-5\nu}{30} - \frac{7(5t^2-3)-20\nu t^2}{5(7+5\nu)}(1-\rho_0^2)\right](\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) \\
& - \left[\frac{\rho_0^2/t^2-1}{2t^3} - \frac{2t^2(7-10\nu)}{7+5\nu}(1-\rho_0^2)\right]\delta_{ij}\bar{x}_m\bar{x}_n - \left[\frac{\rho_0^2/t^2+2\nu-1}{2t^3} - \frac{8\nu t^2}{7+5\nu}(1-\rho_0^2)\right]\delta_{mn}\bar{x}_i\bar{x}_j \\
& - \left[\frac{\rho_0^2/t^2-\nu}{2t^3} + \frac{6\nu t^2}{7+5\nu}(1-\rho_0^2)\right](\delta_{im}\bar{x}_j\bar{x}_n + \delta_{in}\bar{x}_j\bar{x}_m + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m) + \frac{7\rho_0^2/t^2-5}{2t^3}\bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n\Bigg\}
\end{aligned} \tag{110}
$$

Figure 4 shows a comparison of the Neumann–Eshelby tensor with the original Eshelby tensor for $\rho_0=0.4$ and $\nu=0.3$. Here we display the six coefficients of the radial basis arrays of the finite Eshelby tensors, $\mathbf{S}^{\cdot,N}$ and the original Eshelby tensors, $\mathbf{S}^{\cdot,\infty}$. One can see that there are significant differences in the first three coefficients.

A display of the displacement bases, $\mathbf{U}^{\cdot,N}$, $\mathbf{U}^{\cdot,\infty}$, and $\mathbf{U}^{B,N}$ for $\nu=0.3$, is shown in Figs. 3(*b*), 3(*d*), and 3(*f*). One can observe that the difference between the Neumann and the original solution is large, even though the volume fraction is only $\rho_0^3=0.4^3=0.064$. Figure 3 also illustrates different characters of the Dirichlet and the Neumann solution.

Remark 4.1. The volumetric part of the disturbance strain ϵ^d_{ij} is related to the volumetric part of the eigenstrain ϵ^*_{ij} by a scalar coefficient

$$
\mathbb{S}^{\cdot,\star}_{iijj}(\mathbf{x}) = \mathbf{\Theta}_{iijj}\mathbf{S}^{\cdot,\star}(t)
$$

with

$$
\mathbf{\Theta}_{iijj} = [9,6,3,3,4,1]^T \tag{111}
$$

From this, one can find some interesting relationships of the finite Eshelby tensors. First we have

$$
\mathbb{S}^{\cdot,\star}_{iijj}(\mathbf{x}) = 9\mathbf{S}^{\cdot,\star}_1 + 6\mathbf{S}^{\cdot,\star}_2 + 3\mathbf{S}^{\cdot,\star}_3 + 3\mathbf{S}^{\cdot,\star}_4 + 4\mathbf{S}^{\cdot,\star}_5 + \mathbf{S}^{\cdot,\star}_6 = \text{const.}\ \forall\, \mathbf{x} \in \Omega \tag{112}
$$

This implies that even though the finite Eshelby tensors derived here are functions of the position vector $\mathbf{x}$, the dilatational part of the Eshelby tensor is a constant. In particular, the following dilatational contractions have elementary forms,

$$
\mathbb{S}^{I,D}_{iijj} = \frac{(1-f)(1+\nu)}{1-\nu}, \quad \mathbb{S}^{E,D}_{iijj} = -\frac{f(1+\nu)}{1-\nu} \tag{113}
$$

$$
\mathbb{S}^{I,N}_{iijj} = \frac{(1+\nu)+2f(1-2\nu)}{1-\nu}, \quad \mathbb{S}^{E,N}_{iijj} = \frac{2f(1-2\nu)}{1-\nu} \tag{114}
$$

where $f=\rho_0^3$ is the volume fraction of the inclusion phase. Second it is interesting to note that the Dirichlet and the Neumann Eshelby tensors follow the ordering

$$
\mathbb{S}^{I,N}_{iijj} \geqslant \mathbb{S}^{I,D}_{iijj}, \quad \mathbb{S}^{E,N}_{iijj} \geqslant \mathbb{S}^{E,D}_{iijj}, \quad \text{“ = ” holds iff } f=0 \tag{115}
$$

and that the difference between interior and exterior solution is

$$
\mathbb{S}^{I,D}_{iijj} - \mathbb{S}^{E,D}_{iijj} = \mathbb{S}^{I,N}_{iijj} - \mathbb{S}^{E,N}_{iijj} = \frac{1+\nu}{1-\nu} = \mathbb{S}^{I,\infty}_{iijj} \tag{116}
$$

In classical theory, the dilatational eigenstrain has some special properties, e.g., the dilatational eigenstrain due to a dilating inclusion is constant. It appears that some of these properties are still preserved in the finite spherical inclusion solution. This not only validates the present theory, but also indicates that the present theory may have some important applications, because dilatational eigenstrains are usually associated with, for example, thermal expansion, lattice mismatch in quantum dots, and misfit strain in phase transformation.

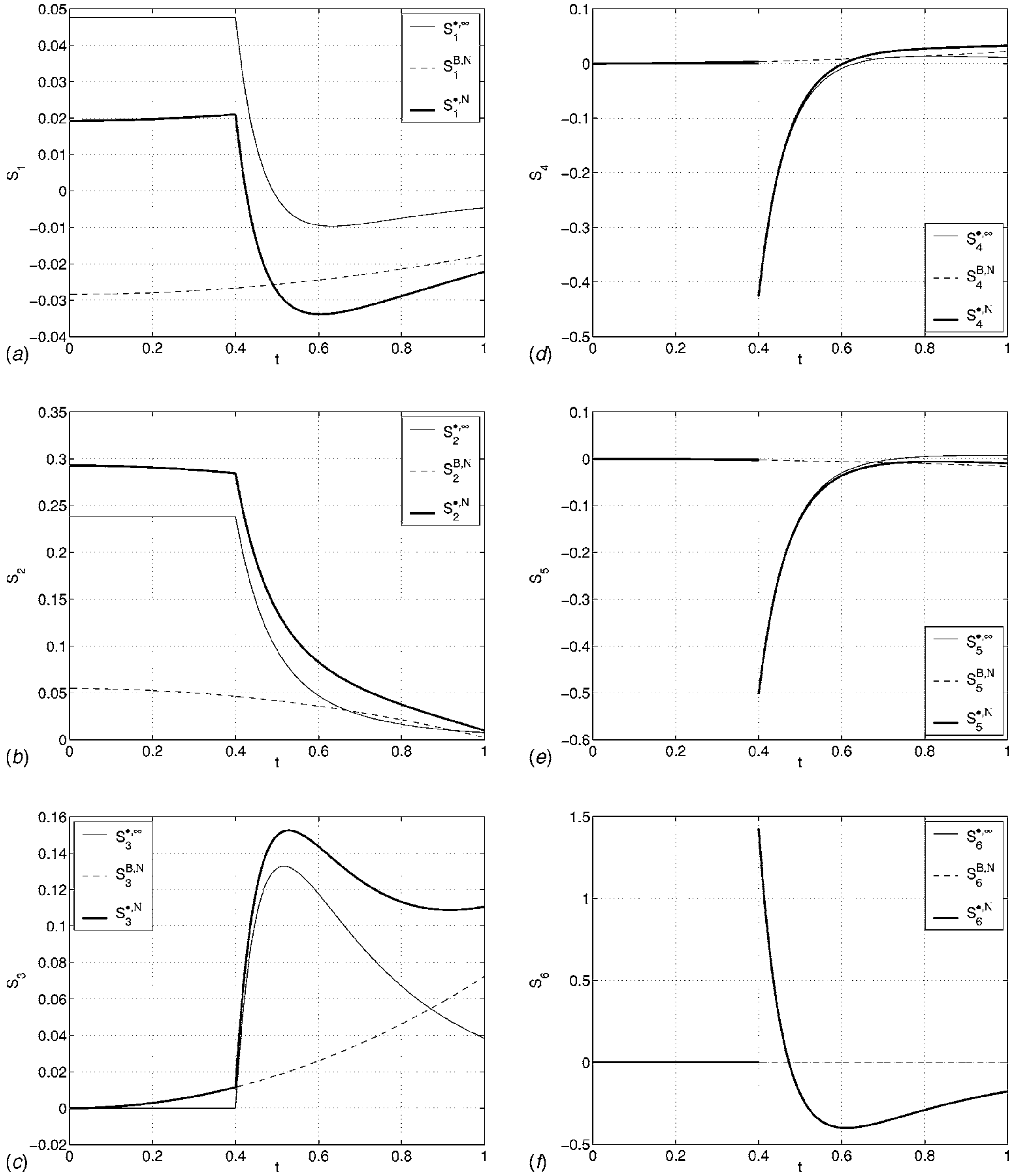

Fig. 4 The components of the radial basis arrays $\mathbf{S}^{\cdot,\infty}$, $\mathbf{S}^{B,N}$, and $\mathbf{S}^{\cdot,N}$

4.3 Traction Distributions. Next, we examine the radial projection of the disturbance stress field. The physical meaning of such a stress projection field is a set of parametric traction fields on the surfaces of successive concentric spheres. Any point, $\mathbf{x}$, inside the spherical RVE lies on a spherical surface whose normal $\bar{\mathbf{x}}$ is along the direction of the position vector $\mathbf{x}$. Thus the parametric traction field is defined as

$$t_i^d(\mathbf{x}) = \sigma_{ji}^d(\mathbf{x})\bar{x}_j(\mathbf{x}) \tag{117}$$

which can be expressed in terms of the eigenstrain

$$t_i^d(\mathbf{x}) = \begin{cases} \bar{x}_j(\mathbf{x})\mathbb{C}_{ijk\ell}[\mathbb{S}_{k\ell mn}^{I,\star}(\mathbf{x}) - \mathbb{I}_{k\ell mn}^s]\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega_I \\ \bar{x}_j(\mathbf{x})\mathbb{C}_{ijk\ell}\mathbb{S}_{k\ell mn}^{E,\star}(\mathbf{x})\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega_E \end{cases} \tag{118}$$

Here $\mathbb{I}_{k\ell mn}^s$ is the fourth-order symmetric identity tensor, which also falls into our definition of a fourth-order radial isotropic tensor, i.e.,

$$\mathbb{I}_{k\ell mn}^s = \boldsymbol{\Theta}_{k\ell mn}^T(\mathbf{r})\mathbf{I}^s \tag{119}$$

where $\mathbf{I}^s = [0, 1/2, 0, 0, 0, 0]^T$. One may then rewrite Eq. (118) as

$$t_i^d(\mathbf{x}) = \begin{cases} \bar{x}_j(\mathbf{x})\mathbb{C}_{ijk\ell}\boldsymbol{\Theta}_{k\ell mn}^T(\bar{\mathbf{x}})[\mathbf{S}^{I,\star}(t) - \mathbf{I}^s)\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega_I \\ \bar{x}_j(\mathbf{x})\mathbb{C}_{ijk\ell}\boldsymbol{\Theta}_{k\ell mn}^T(\bar{\mathbf{x}})\mathbf{S}^{E,\star}(t)\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega_E \end{cases} \tag{120}$$

In analogy to the displacement field (see Eq. (43)) the disturbance traction can also be written as

$$t_i^d(\mathbf{x}) = \boldsymbol{\Xi}_{imn}^T(\bar{\mathbf{x}})\mathbf{T}^{\cdot,\star}(t)\epsilon_{mn}^* \tag{121}$$

where $\mathbf{T}^{\cdot,\star}$ is the radial basis array of the traction field and $\boldsymbol{\Xi}_{imn}$ is given by Eq. (42). The preceding two equations establishes a relation between the arrays $\mathbf{T}^{\cdot,\star}$ and $\mathbf{S}^{\cdot,\star}$. We find that

$$\mathbf{T}^{I,\star}(t) = \mathbf{K}_1[\mathbf{S}^{I,\star}(t) - \mathbf{I}^s] \tag{122}$$

$$\mathbf{T}^{E,\star}(t) = \mathbf{K}_1\mathbf{S}^{E,\star}(t) \tag{123}$$

where $\mathbf{K}_1$ is given by Eq. (54). In view of Eqs. (71) and (72) we can write

$$\mathbf{T}^{I,\star}(t) = \mathbf{T}^{I,\infty}(t) + \mathbf{T}^{B,\star}(t) - \mathbf{T}^{*}(t), \quad 0 \leqslant t < \rho_0 \tag{124}$$

$$\mathbf{T}^{E,\star}(t) = \mathbf{T}^{E,\infty}(t) + \mathbf{T}^{B,\star}(t), \quad \rho_0 \leqslant t \leqslant 1 \tag{125}$$

where the individual pieces are as follows. Corresponding to the original Eshelby problem we have

$$\mathbf{T}^{I,\infty}(t) = \mathbf{K}_1\mathbf{S}^{I,\infty}(t) = \frac{2\mu}{15(1-\nu)}\begin{bmatrix}(1-12\nu+5\nu^2)/(2\nu-1)\\ 4-5\nu \\ 0\end{bmatrix} \tag{126}$$

$$\mathbf{T}^{E,\infty}(t) = \mathbf{K}_1\mathbf{S}^{E,\infty}(t) = \frac{\mu\rho_0^3/t^3}{15(1-\nu)}\begin{bmatrix}-12\rho_0^2/t^2+10(1-\nu)\\ -12\rho_0^2/t^2+5(1+\nu) \\ 60(\rho_0^2/t^2-1)\end{bmatrix} \tag{127}$$

and the Dirichlet and Neumann boundary contributions are

$$\mathbf{T}^{B,D}(t) = \mathbf{K}_1\mathbf{S}^{B,D}(t) = -\frac{2\mu\rho_0^3}{15(1-\nu)}\begin{bmatrix}(1-12\nu+5\nu^2)/(2\nu-1)\\ 4-5\nu \\ 0\end{bmatrix} + \frac{\mu\rho_0^3(1-\rho_0^2)}{10(1-\nu)(7-10\nu)}\begin{bmatrix}2(7+5\nu t^2)\\ 7(5t^2-3)+10\nu t^2 \\ 10t^2(7-5\nu)\end{bmatrix} \tag{128}$$

$$\mathbf{T}^{B,N}(t) = \mathbf{K}_1\mathbf{S}^{B,N}(t) = -\frac{\mu\rho_0^3}{15(1-\nu)}\begin{bmatrix}2(1+5\nu)\\ 7-5\nu \\ 0\end{bmatrix} - \frac{2\mu\rho_0^3(1-\rho_0^2)}{5(1-\nu)(7+5\nu)}\begin{bmatrix}2(7+5\nu t^2)\\ 7(5t^2-3)+10\nu t^2 \\ 10t^2(7-5\nu)\end{bmatrix} \tag{129}$$

The final contribution, arising from the eigenstrains, is

$$\mathbf{T}^{*}(t) = \mathbf{K}_1\mathbf{I}^{s} = \begin{bmatrix}2\nu/(1-2\nu)\\ 1 \\ 0\end{bmatrix} \tag{130}$$

It is readily verified that for $t=1$ the traction basis corresponding to the Neumann–Eshelby problem is

$$\mathbf{T}^{E,N}(1) = \begin{bmatrix}0\\ 0 \\ 0\end{bmatrix} \tag{131}$$

which assures $t_i(\mathbf{x}) \equiv 0$ for $\forall \mathbf{x} \in \partial\Omega$. Therefore the prescribed Neumann boundary condition is indeed satisfied by the solution presented. This fact can also be clearly observed in Fig. 5, which shows the three components of $\mathbf{T}^{\cdot,\infty}$, $\mathbf{T}^{\cdot,\star}$ and $\mathbf{T}^{B,\star}$ for both the Dirichlet problem (a),(c),(e), and the Neumann problem (b),(d),(f). Here we choose $\rho_0=0.4$ and $\nu=0.3$. We observe that the components of $\mathbf{T}^{E,N}(t)$ go to zero at the boundary of the RVE ($t=1$). It can also be seen that the boundary corrections $\mathbf{T}^{B,D}$ and $\mathbf{T}^{B,N}$ are substantial even though the volume fraction is small, i.e., $\rho_0^3=0.064$.

5 Closure

In this paper, the elastic fields due to a spherical inclusion subjected to prescribed eigenstrains and embedded in a finite spherical RVE are studied. On the outer surface of the RVE, uniform boundary conditions are prescribed, which are either a displacement (Dirichlet) boundary condition or a prescribed traction (Neumann) boundary condition.

The notion of a *radial isotropic tensor* is introduced, which is a generalization of the isotropic tensor. It has been argued that if a spherical inclusion is placed concentrically within a spherical RVE, the finite Eshelby tensors, which map the prescribed eigenstrain to the disturbance strain field, are radial isotropic tensors. In other words, the tensorial circumference basis for the finite Eshelby tensors is the same as the basis for the Eshelby tensors in unbounded space.

By utilizing this property, we have solved a pair of Fredholm type integral equations, and we have obtained, for the first time, the exact, closed form solutions for both the interior and exterior Eshelby tensors for an inclusion in a finite, three-dimensional RVE. It has been revealed that the finite Eshelby tensors depend on both the location and the volume fraction of the inclusion, which accurately captures both the size effect of the inclusion and the boundary image contribution to the original inclusion problem.

One of advantages of the present solution procedure is that it circumvents the use of a finite Green's function. As a matter of fact, the solution of Green's function of Navier's equation for a finite spherical domain is a more difficult problem, which is still open. On the other hand, we hope that this work may shed some light on the search for the finite Green's function, however, we believe, not without some added difficulties. We further note that, by using our solution technique, one may be able to extend the present solution to the elliptical inclusion problem in a finite domain. The difficulty then will be how to find the symmetry group of the circumference basis of the elliptical geometry, which has to be also invariant under the integral equation that involves the boundary integrals Eqs. (51) and (85).

We also would like to mention that the spherical RVE may be subjected to general boundary conditions. Nevertheless, the two fundamental solutions corresponding to the Dirichlet and the Neumann boundary conditions form a basis for the finite Eshelby tensors under a general boundary value problem. This issue will be further discussed in detail in a separate paper [16]. It should also be pointed out that even though the two basic finite Eshelby tensors obtained here are the solutions of the homogeneous inclusion problems, they are two fundamental elements for the finite Eshelby tensors of a general RVE with more complex microstructures. By using superposition, they can be readily used to construct the solutions for the n-inclusion ($n \geqslant 2$) problem, and they can be used to solve various homogenization problems as well as the problem of inhomogeneity induced elastic fields in a finite spherical domain.

To illustrate such applications, in the second part of this work [11], we apply the finite Eshelby tensors to evaluate the effective material properties of composites. It has been shown that the method employing the finite Eshelby tensor provides remarkably accurate predictions in simple homogenization procedures. Furthermore they furnish new variational bounds, and lead to a new class of general homogenization methods.

Acknowledgment

This work is supported by a grant from National Science Foundation (Grant No. CMS-0239130), which is greatly appreciated.

Appendix: Integration Formulas

In this Appendix the solution of the fourteen integrals listed in Eqs. (55)–(61) and (90)–(96) is given. The procedure is similar to the two dimensional case reported in Li et al. [17] and Wang et al. [18]. Considering $\mathbf{x}+r\overline{\mathbf{r}}=\mathbf{y}$, where $\mathbf{y} \in \partial\Omega$, we have

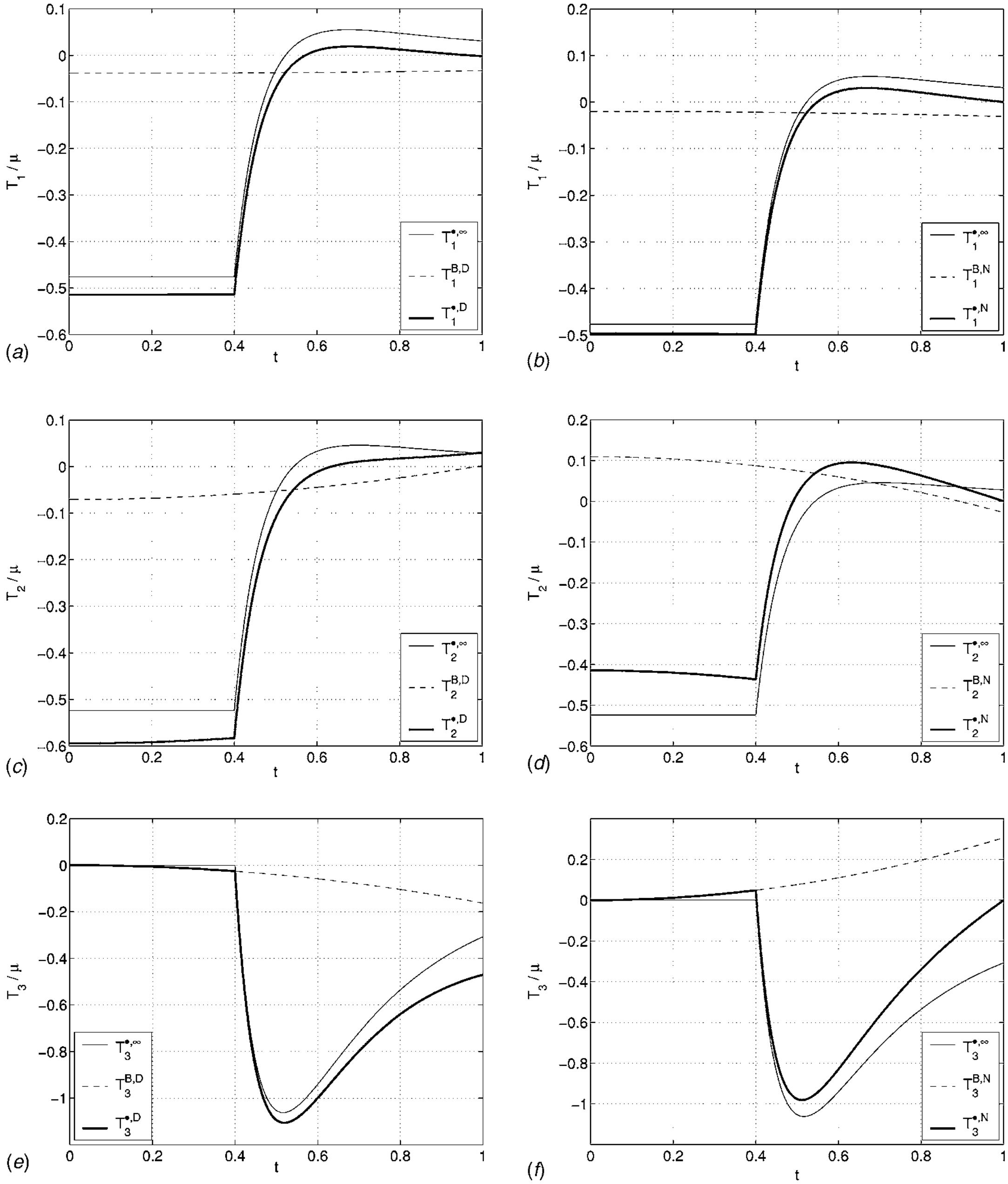

Fig. 5 The components of the radial basis arrays $\mathbf{T}^{\cdot,\infty}$, $\mathbf{T}^{B,D}$, $\mathbf{T}^{\cdot,D}$, $\mathbf{T}^{B,N}$, and $\mathbf{T}^{\cdot,N}$

$$\bar{r}_i = \frac{A}{r}(n_i - t\bar{x}_i) \tag{A1}$$

or

$$n_i = \frac{r}{A}\bar{r}_i + t\bar{x}_i.$$

Recall that $t=|\mathbf{x}|/A$. The relations defined in Eq. (A1) are illustrated in both Figs. 1 and 6(a).

The surface integration over the RVE is performed w.r.t. the surface of a unit sphere, S_2, centered at point $\mathbf{x}$. According to Fig. 6, we define a new basis $\hat{\mathbf{e}}_i$ at $\mathbf{x}$ such that $\hat{\mathbf{e}}_3=\bar{\mathbf{x}}$ Vector $\bar{\mathbf{r}}$ is then described by the spherical coordinates φ and θ, i.e., $\bar{\mathbf{r}} = (\cos\varphi \sin\theta \sin\varphi \sin\theta \cos\theta)^T$.

Denote dS as the surface element of $\partial\Omega$ (the outer surface of the RVE). The projection of dS to the perpendicular direction of $\bar{\mathbf{r}}$ is denoted by $d\hat{S}$, and is given by $d\hat{S}=r^2 \sin\theta \, d\theta \, d\varphi$. It is related to dS by

$$dS = \frac{d\hat{S}}{\cos\psi} = \frac{r^2}{\cos\psi}\sin\theta d\theta d\varphi = \frac{r^2}{\cos\psi}dS_2 \tag{A2}$$

where $dS_2=\sin\theta \, d\theta \, d\varphi$ is the surface element on the unit sphere S_2. Considering the shaded triangle (**0xy**) in Fig. 6, we find that

$$\frac{A}{r} = \frac{1}{\sqrt{1-2t\cos\phi + t^2}} \tag{A3}$$

and

$$\cos\psi = \sqrt{1-t^2\sin^2\theta} \tag{A4}$$

Furthermore from $y_i y_i = A^2$, one can derive the relation

$$r = A(-t\cos\theta + \sqrt{1-t^2\sin^2\theta}) \tag{A5}$$

Figure 6(b) shows that for every point P on the surface of the unit sphere there exists a point P^* such that $\bar{\mathbf{r}}(P)=-\bar{\mathbf{r}}(P^*)$. Thus any function, $\mathcal{L}^o(\bar{\mathbf{r}})=\bar{r}_i, \bar{r}_i\bar{r}_j, \bar{r}_i\bar{r}_j\bar{r}_m, \ldots$, which is odd in $\bar{\mathbf{r}}$, satisfies

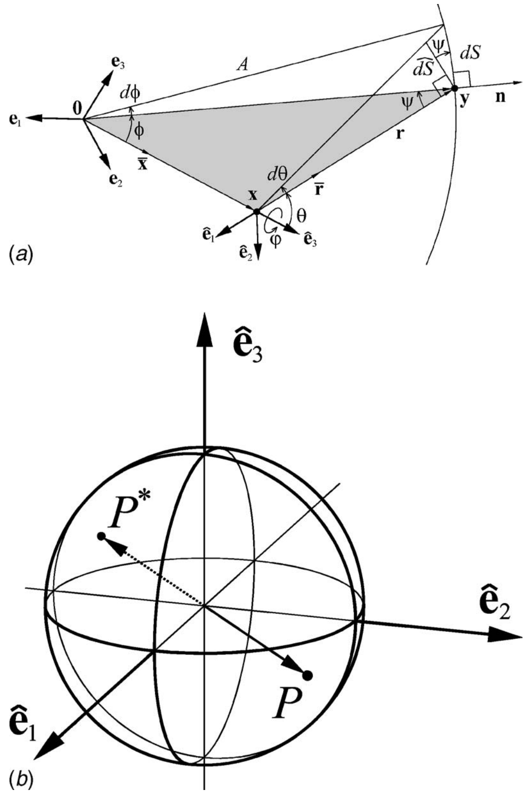

Fig. 6 (*a*) Relation between *dS*, $d\phi$, and $d\theta$; and (*b*) unit sphere.

$\mathcal{L}^o[\overline{\mathbf{r}}(P)] = -\mathcal{L}^o[\overline{\mathbf{r}}(P^*)]$, and therefore the integration of an odd function of $\overline{\mathbf{r}}$, i.e., $\mathcal{L}^o(\overline{\mathbf{r}})$, over the surface of the sphere will be zero. In particular, by applying Eqs. (A2) and (A4), we find that

$$\int_{\partial\Omega} \frac{\mathcal{L}^o(\overline{\mathbf{r}})}{r^2} dS = \int_{S_2} A \frac{\mathcal{L}^o(\overline{\mathbf{r}})}{\sqrt{1 - t^2 \sin^2\theta}} dS_2 = 0 \tag{A6}$$

Note that $\sin^2\theta$ is an even function in $\overline{\mathbf{r}}$, i.e., $\sin^2\theta(\overline{\mathbf{r}}) = \sin^2\theta(-\overline{\mathbf{r}})$. Further, we denote an even function of $\overline{\mathbf{r}}$ as $\mathcal{L}^e(\overline{\mathbf{r}})$, if $\mathcal{L}^e[\overline{\mathbf{r}}(P)] = \mathcal{L}^e[\overline{\mathbf{r}}(P^*)]$. Then, by virtue of Eqs. (A3)–(A5), it follows that

$$\int_{\partial\Omega} \frac{\mathcal{L}^e(\overline{\mathbf{r}})}{r} dS = \int_{S_2} A\left(1 - \frac{t\cos\theta}{\sqrt{1 - t^2\sin^2\theta}}\right)\mathcal{L}^e(\overline{\mathbf{r}}) dS_2 = A\int_{S_2} \mathcal{L}^e(\overline{\mathbf{r}}) dS_2 \tag{A7}$$

because $\cos\theta$ is an odd function in $\overline{\mathbf{r}}$. Using Eqs. (A7) and (A6) we obtain the following seven elemental integrals

$$(1) \quad \int_{\partial\Omega} \frac{1}{r} dS_y = 4\pi A \tag{A8}$$

$$(2) \quad \int_{\partial\Omega} \frac{\overline{r}_i}{r^2} dS_y = 0 \tag{A9}$$

$$(3) \quad \int_{\partial\Omega} \frac{\overline{r}_i\overline{r}_j}{r} dS_y = A\int_{S_2} \overline{r}_i\overline{r}_j \, dS_u = \frac{4\pi}{3}A\delta_{ij} \tag{A10}$$

$$(4) \quad \int_{\partial\Omega} \frac{\overline{r}_i\overline{r}_j\overline{r}_m}{r^2} dS_y = 0 \tag{A11}$$

$$(5) \quad \int_{\partial\Omega} \frac{\overline{r}_i\overline{r}_j\overline{r}_m\overline{r}_n}{r} dS_y = A\int_{S_2} \overline{r}_i\overline{r}_j\overline{r}_m\overline{r}_n \, dS_u = \frac{4\pi}{15}A(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{A12}$$

$$(6) \quad \int_{\partial\Omega} \frac{\overline{r}_i\overline{r}_j\overline{r}_m\overline{r}_n\overline{r}_r}{r^2} dS_y = 0 \tag{A13}$$

$$\begin{aligned}(7) \quad \int_{\partial\Omega} \frac{\overline{r}_i\overline{r}_j\overline{r}_m\overline{r}_n\overline{r}_r\overline{r}_s}{r} dS_y = A\int_{S_2} \overline{r}_i\overline{r}_j\overline{r}_m\overline{r}_n\overline{r}_r\overline{r}_s \, dS_u = \frac{4\pi}{105}A(&\delta_{ij}\delta_{mn}\delta_{rs} \\ &+ \delta_{im}\delta_{jn}\delta_{rs} + \delta_{in}\delta_{jm}\delta_{rs} + \delta_{ir}\delta_{mn}\delta_{js} \\ &+ \delta_{is}\delta_{mn}\delta_{jr}\delta_{ij}\delta_{mr}\delta_{ns} + \delta_{im}\delta_{jr}\delta_{ns} \\ &+ \delta_{in}\delta_{jr}\delta_{ms} + \delta_{ir}\delta_{mj}\delta_{ns} \\ &+ \delta_{is}\delta_{mj}\delta_{nr}\delta_{ij}\delta_{ms}\delta_{nr} + \delta_{im}\delta_{js}\delta_{nr} \\ &+ \delta_{in}\delta_{js}\delta_{mr} + \delta_{ir}\delta_{ms}\delta_{nj} + \delta_{is}\delta_{mr}\delta_{nj})\end{aligned} \tag{A14}$$

Using these seven elemental integrals and Eq. (A1) we obtain all the integrals listed in Eqs. (55)–(61) and (90)–(96).

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