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A circular inclusion in a finite domain II. The Neumann-Eshelby problem

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Summary. This is the second paper in a series concerned with the precise characterization of the elastic fields due to inclusions embedded in a finite elastic medium. In this part, an exact and closed form solution is obtained for the elastic fields of a circular inclusion embedded in a finite circular representative volume element (RVE), which is subjected to the traction (Neumann) boundary condition. The disturbance strain field due to the presence of an inclusion is related to the uniform eigenstrain field inside the inclusion by the so-called Neumann-Eshelby tensor. Remarkably, an elementary, closed form expression for the Neumann-Eshelby tensor of a circular RVE is obtained in terms of the volume fraction of the inclusion. The newly derived Neumann-Eshelby tensor is complementary to the Dirichlet-Eshelby tensor obtained in the first part of this work. Applications of the Neumann-Eshelby tensor are discussed briefly.

1 Introduction

In part I of this serial work, a novel solution procedure is developed to solve a class of Fredholm integral equations. By applying this technique, an exact solution of the elastic fields due to a circular inclusion inside a finite domain is obtained, which is subjected to prescribed displacement (Dirichlet) boundary conditions. This second paper deals with its dual problem: the elastic fields due to a circular inclusion inside a finite circular representative volume element (RVE) that is purely subjected to prescribed traction (Neumann) boundary conditions.

In finite domains, the Eshelby tensors are expected to form a duality pair for two different boundary conditions, much like the Hashin-Shtrikman variational bounds (Hashin and Shtrikman [3], [4], Hill [5], and Weng [9]) come as a pair from minimization of elastic potential energy and complementary potential energy. The original Eshelby tensors (Eshelby [1], [2]) for an ellipsoidal inclusion are obtained in an unbounded domain, and there is no boundary condition effect. This is because an infinite space has no boundary, and therefore the boundary effect never affects the inclusion solution. The Eshelby tensor for an infinite domain serves as a good approximation if the size of an inclusion is small. However, in the ensuing homogenization processes, effects of different boundary conditions become important, especially in the range of moderate to high inclusion concentrations. In engineering applications, all RVEs are finite, and the Eshelby tensors become domain dependent. As discovered in Part I (Li et al. [6]) of this series, the Dirichlet-Eshelby tensor depends on the ratio between the size of the inclusion and the size of the representative volume element (RVE).

2 Inclusion problem

Consider a circular inclusion Ω_e embedded at the center of a circular representative volume element Ω. The following traction boundary condition is prescribed on the boundary of the RVE,

$$\sigma_{ji}(\mathbf{x})n_j = \bar{t}_i = \sigma^0_{ji}n_j, \quad \forall \mathbf{x} \in \partial\Omega, \tag{1}$$

where σ^0_{ij} is a constant stress tensor.

Under the pure traction boundary condition, Hill [3] showed that the remote constant stress tensor is equal to the average stress in the RVE, i.e.,

$$\sigma^0_{ij} = \langle \sigma_{ij}(\mathbf{x}) \rangle_\Omega =: \bar{\sigma}_{ij}, \quad \forall \mathbf{x} \in \Omega. \tag{2}$$

The total stress field may be decomposed into two parts: the average stress field (also the remote stress, macro-stress) and the disturbance stress field,

$$\sigma_{ij}(\mathbf{x}) = \bar{\sigma}_{ij} + \sigma^d_{ij}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega. \tag{3}$$

From the average theorem (2), the average of the disturbance stress field vanishes, i.e. $\langle \sigma^d_{ij}(\mathbf{x}) \rangle_\Omega = 0$; and the induced disturbance traction field vanishes on the prescribed traction boundary,

$$\sigma^d_{ji}(\mathbf{x})n_j = 0, \quad \forall \mathbf{x} \in \partial\Omega. \tag{4}$$

The total strain field may be viewed as the superposition of the induced remote strain field and the disturbance strain field,

$$\epsilon_{ij}(\mathbf{x}) = \epsilon^0_{ij} + \epsilon^d_{ij}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{5}$$

where the induced constant strain field on the remote RVE boundary is defined as

$$\epsilon^0_{ij} =: \mathbb{D}_{ijk\ell}\sigma^0_{k\ell} = \mathbb{D}_{ijk\ell}\bar{\sigma}_{k\ell}, \tag{6}$$

and $\mathbb{D}_{ijk\ell}$ is the elastic compliance tensor of the matrix.

In the presence of the second phase inclusion inside the RVE, the average strain field is not equal to the remote strain of the comparison matrix, i.e., $\epsilon^0_{ij} \neq \langle \epsilon_{ij}(\mathbf{x}) \rangle_\Omega =: \bar{\epsilon}_{ij}$, which means that the average of the disturbance strain field will not be zero in this case.

To account for the misfit of an inclusion, a piecewise constant eigenstrain field is prescribed inside the RVE, i.e.,

$$\epsilon^*_{ij}(\mathbf{x}) = \begin{cases} \epsilon^*_{ij}, & \forall \mathbf{x} \in \Omega_e, \\ 0, & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{7}$$

In terms of the eigenstrain field, we have

$$\sigma_{ij}(\mathbf{x}) = \sigma^0_{ij} + \sigma^d_{ij}(\mathbf{x}) = \begin{cases} \mathbb{C}_{ijk\ell}\big(\epsilon^0_{k\ell} + \epsilon^d_{k\ell}(\mathbf{x}) - \epsilon^*_{k\ell}\big), & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{C}_{ijk\ell}\big(\epsilon^0_{k\ell} + \epsilon^d_{k\ell}(\mathbf{x})\big), & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{8}$$

$$\epsilon_{ij}(\mathbf{x}) = \epsilon^0_{ij} + \epsilon^d_{ij}(\mathbf{x}) = \begin{cases} \mathbb{D}_{ijk\ell}\big(\sigma^0_{k\ell} + \sigma^d_{k\ell}(\mathbf{x})\big) + \epsilon^*_{ij}, & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{D}_{ijk\ell}\big(\sigma^0_{k\ell} + \sigma^d_{k\ell}(\mathbf{x})\big), & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{9}$$

Equivalently, the disturbance strain and stress fields are related via

$$\epsilon^d_{ij}(\mathbf{x}) = \mathbb{D}_{ijk\ell}\sigma^d_{k\ell}(\mathbf{x}) + \epsilon^*_{k\ell}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{10}$$

or

$$\sigma_{ij}^{d}(\mathbf{x}) = \begin{cases} \mathbb{C}_{ijk\ell}(\epsilon_{k\ell}^{d}(\mathbf{x}) - \epsilon_{k\ell}^{*}), & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{C}_{ijk\ell}\epsilon_{k\ell}^{d}(\mathbf{x}), & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{11}$$

Analogously to the Eshelby tensor for an infinite space, the interior/exterior Neumann-Eshelby tensors $\left(\mathbb{S}_{ijmn}^{I,N}, \mathbb{S}_{ijmn}^{E,N}\right)$ for a finite domain are introduced to characterize the disturbance strain field in terms of the prescribed eigenstrain for a finite RVE under the Neumann boundary condition, such that

$$\epsilon_{ij}^{d}(\mathbf{x}) = \begin{cases} \mathbb{S}_{ijmn}^{I,N}(\mathbf{x})\epsilon_{mn}^{*}, & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{S}_{ijmn}^{E,N}(\mathbf{x})\epsilon_{mn}^{*}, & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{12}$$

Assume that the solid is linear elastic,

$$\sigma_{ij}(\mathbf{x}) = \mathbb{C}_{ijk\ell}e_{k\ell}(\mathbf{x}), \tag{13}$$

where $\mathbb{C}_{ijk\ell}$ is the elastic tensor and the infinitesimal elastic strain is defined as

$$e_{ij}(\mathbf{x}) = \epsilon_{ij}(\mathbf{x}) - \epsilon_{ij}^{*}(\mathbf{x}). \tag{14}$$

The equilibrium equation, $\sigma_{ji,j}(\mathbf{x}) = 0$, leads to the following boundary value problem:

$$\mathbb{C}_{ijk\ell}u_{k,\ell j}^{d}(\mathbf{x}) - \mathbb{C}_{ijk\ell}\epsilon_{k\ell,j}^{*}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \Omega, \tag{15}$$

$$\sigma_{ji}^{d}(\mathbf{x})n_j(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\Omega. \tag{16}$$

It is noted that the eigenstrain distribution is localized within the inclusion, and it is not present on the RVE boundary.

Denote Green's function, $G_{mk}^{\infty}(\mathbf{x}-\mathbf{y})$, as the solution of the following Navier's equation for an infinite elastic domain in a two-dimensional space,

$$\mathbb{C}_{ijk\ell}G_{mk,\ell j}^{\infty}(\mathbf{x}-\mathbf{y}) + \delta_{mi}\delta(\mathbf{x}-\mathbf{y}) = 0, \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^2, \tag{17}$$

which gives

$$G_{ij}^{\infty}(\mathbf{x}-\mathbf{y}) = \frac{1}{8\pi\mu(1-\nu)}\left\{\ell_i\ell_j - (3-4\nu)\delta_{ij}\ln R\right\}, \tag{18}$$

where ν is Poisson's ratio, μ is the shear modulus, $\ell_i = (y_i - x_i)/R$, and $R = |\mathbf{y}-\mathbf{x}|$.

As in Part I of this work, the circular RVE depicted in Fig.1 has radius H_0, with a circular inclusion of radius a in the center. The ratio of inclusion and RVE is characterized by the dimensionless parameter $\rho_0 = a/H_0$. The normalized radial position for an arbitrary vector $\mathbf{x} \in \Omega$ is denoted as $t = |\mathbf{x}|/H_0$, and its circumference variation is characterized by a unit normal vector $\mathbf{r}$ defined as $\mathbf{r}(\mathbf{x}) =: \mathbf{x}/|\mathbf{x}|$. For clarity, we reserve the symbol $\mathbf{n}(\mathbf{y}) =: \mathbf{y}/|\mathbf{y}|$ if $\mathbf{y} \in \partial\Omega$ to emphasize its position on the RVE boundary. The argument $\mathbf{x}$ or $\mathbf{y}$ may be dropped in the following if no confusion can occur.

Using Somigliana's identity [8] and considering the prescribed traction boundary condition (16), the disturbance displacement field may be written as the integral representation:

$$u_m^{d}(\mathbf{x}) = \oint_{\partial\Omega}\mathbb{C}_{ijk\ell}u_k^{d}(\mathbf{y})G_{im,j}^{\infty}(\mathbf{x}-\mathbf{y})n_\ell(\mathbf{y})dS_y - \int_{\Omega}\mathbb{C}_{ijk\ell}G_{im,j}^{\infty}(\mathbf{x}-\mathbf{y})\epsilon_{k\ell}^{*}(\mathbf{y})d\Omega_y. \tag{19}$$

It is noted that the first part of Eq. (19) is integrated along $\mathbf{y} \in \partial\Omega$. When evaluated at $\mathbf{x} \in \partial\Omega$, it becomes a weakly singular integral equation. To avoid discussions on weakly singular integral equations, we restrict $\mathbf{x} \in \mathring{\Omega}$ for the moment, which means that $\mathbf{x}$ is in the interior region of the domain Ω.

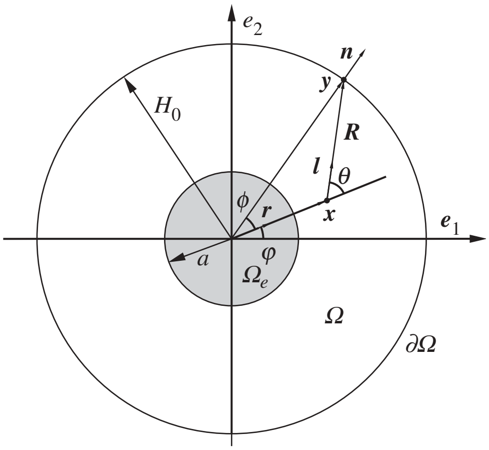

Fig. 1. A circular representative element containing a circular inclusion

A similar type of integral equation was solved in Part I. However, the prescribed traction (Neumann) boundary value problem is fundamentally different from the prescribed displacement (Dirichlet) boundary value problem. Instead of solving a Fredholm type integral equation for the strain field, a Fredholm type integral equation for the displacement field $\mathbf{u}^d$ has to be solved, i.e.,

$$u_m^d(\mathbf{x}) = \oint_{\partial\Omega} \mathbb{C}_{ijk\ell} u_k^d(\mathbf{y}) G_{im,j}^{\infty}(\mathbf{x}-\mathbf{y}) n_\ell(\mathbf{y}) dS_y - \epsilon_{k\ell}^* \int_{\Omega_e} \mathbb{C}_{ijk\ell} G_{im,j}^{\infty}(\mathbf{x}-\mathbf{y}) d\Omega_y. \tag{20}$$

A new third-order tensor, $\mathbb{U}_{imn}^{\bullet,N}(\mathbf{x})$, is introduced to characterize the disturbance displacement field in terms of the prescribed eigenstrain

$$u_i^d(\mathbf{x}) = \begin{cases} \mathbb{U}_{imn}^{I,N}(\mathbf{x})\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{U}_{imn}^{E,N}(\mathbf{x})\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega/\Omega_e, \end{cases} \tag{21}$$

such that the expected strain field solution can be obtained as Eqs. (12), i.e.,

$$\begin{aligned} \epsilon_{ij}^d(\mathbf{x}) &= \frac{1}{2}\left(u_{i,j}^d(\mathbf{x}) + u_{j,i}^d(\mathbf{x})\right) \\ &= \begin{cases} \frac{1}{2}\left(\mathbb{U}_{imn,j}^{I,N}(\mathbf{x}) + \mathbb{U}_{jmn,i}^{I,N}(\mathbf{x})\right)\epsilon_{mn}^* = \mathbb{S}_{ijmn}^{I,N}(\mathbf{x})\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega_e, \\ \frac{1}{2}\left(\mathbb{U}_{imn,j}^{E,N}(\mathbf{x}) + \mathbb{U}_{jmn,i}^{E,N}(\mathbf{x})\right)\epsilon_{mn}^* = \mathbb{S}_{ijmn}^{E,N}(\mathbf{x})\epsilon_{mn}^*, & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \end{aligned} \tag{22}$$

As stated in Part I, the main hypothesis of our approach is that the Eshelby tensors for a circular inclusion in a circular RVE are "*radial isotropic fourth-order tensors*", i.e., they can be decomposed into a radial basis and a circumference basis. Using the matrix product of two one-dimensional (1D) arrays, we may express the components of the fourth-order interior and exterior Eshelby tensors in the following compact form:

$$\mathbb{S}_{ijmn}^{I,N}(\mathbf{x}) = \boldsymbol{\Theta}_{ijmn}^T(\mathbf{r})\mathbf{S}^{I,N}(t), \tag{23}$$

$$\mathbb{S}_{ijmn}^{E,N}(\mathbf{x}) = \boldsymbol{\Theta}_{ijmn}^T(\mathbf{r})\mathbf{S}^{E,N}(t), \tag{24}$$

where the 1D arrays, $\mathbf{S}^{I,N}(t)$ and $\mathbf{S}^{E,N}(t)$, are specified as

$$\mathbf{S}^{I,N}(t)=\begin{bmatrix}S_1^{I,N}(t)\\S_2^{I,N}(t)\\S_3^{I,N}(t)\\S_4^{I,N}(t)\\S_5^{I,N}(t)\end{bmatrix}\quad\text{and}\quad\mathbf{S}^{E,N}(t)=\begin{bmatrix}S_1^{E,N}(t)\\S_2^{E,N}(t)\\S_3^{E,N}(t)\\S_4^{E,N}(t)\\S_5^{E,N}(t)\end{bmatrix},\tag{25}$$

and the vector $\Theta_{ijmn}(\mathbf{r})$ is the circumference basis of the Eshelby tensors, which is defined as

$$\Theta_{ijmn}(\mathbf{r}):=\begin{bmatrix}\delta_{ij}\delta_{mn}\\\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}\\\delta_{ij}r_mr_n\\r_ir_j\delta_{mn}\\r_ir_jr_mr_n\end{bmatrix}.\tag{26}$$

Similarly, the tensor $\mathbb{U}_{imn}^{\bullet,N}(\mathbf{x})$ is a "*third-order radial isotropic tensor*", and it can only admit the following form, which may be represented by the inner product of two 1D arrays:

$$\mathbb{U}_{imn}^{I,N}(\mathbf{x})=\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{I,N}(t),\quad\forall\mathbf{x}\in\Omega_e,\tag{27}$$

$$\mathbb{U}_{imn}^{E,N}(\mathbf{x})=\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{E,N}(t),\quad\forall\mathbf{x}\in\Omega/\Omega_e,\tag{28}$$

where the 1D arrays are defined as

$$\mathbf{U}^{I,N}(t)=\begin{bmatrix}U_1^{I,N}(t)\\U_2^{I,N}(t)\\U_3^{I,N}(t)\end{bmatrix},\quad\mathbf{U}^{E,N}(t)=\begin{bmatrix}U_1^{E,N}(t)\\U_2^{E,N}(t)\\U_3^{E,N}(t)\end{bmatrix}\quad\text{and}\quad\Xi_{imn}(\mathbf{r})=\begin{bmatrix}r_i\delta_{mn}\\r_m\delta_{in}+r_n\delta_{im}\\r_ir_mr_n\end{bmatrix}.\tag{29}$$

So the disturbance displacement field decomposes into a radial and circumference basis:

$$u_i^d(\mathbf{x})=u_i^d(t,\mathbf{r})=\epsilon_{mn}^*\;\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{I,N}(t),\quad\forall\mathbf{x}\in\Omega_e,\tag{30}$$

$$u_i^d(\mathbf{x})=u_i^d(t,\mathbf{r})=\epsilon_{mn}^*\;\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{E,N}(t),\quad\forall\mathbf{x}\in\Omega/\Omega_e.\tag{31}$$

Further, the kinematic relation (22) yields the following differential mapping that uniquely determines the radial basis of the strain and the displacement fields via

$$\mathbf{S}^{I,N}(t)=\mathfrak{D}(t)\mathbf{U}^{I,N}(t)\quad\text{and}\quad\mathbf{S}^{E,N}(t)=\mathfrak{D}(t)\mathbf{U}^{E,N}(t),\tag{32}$$

where $\mathfrak{D}(t)$ is a differential operator defined in matrix form as

$$\mathfrak{D}(t)=\frac{1}{H_0}\begin{bmatrix}\frac{1}{t}&\frac{1}{t}-\frac{d}{dt}&-\frac{1}{t}\\0&\frac{1}{2}\left(\frac{1}{t}+\frac{d}{dt}\right)&\frac{1}{2t}\\0&-\frac{1}{t}+\frac{d}{dt}&\frac{2}{t}\\-\frac{1}{t}+\frac{d}{dt}&-\frac{1}{t}+\frac{d}{dt}&\frac{1}{t}\\0&0&-\frac{3}{t}+\frac{d}{dt}\end{bmatrix}^{3\times5}.\tag{33}$$

3 Solution of the integral equation

To solve the Fredholm integral equation (20) in the form of Eqs. (30) and (31), we substitute

$$u_k^d(\mathbf{y}) = \epsilon_{mn}^* \Xi_{kmn}^T(\mathbf{n_y})\mathbf{U}^{E,N}(1), \quad \forall \mathbf{y} \in \partial\Omega \tag{34}$$

into Eq. (20), which yields

$$\begin{aligned} u_i^d(\mathbf{x}) = &-\epsilon_{mn}^* \int_{\Omega_e} \mathbb{C}_{pqmn} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y}) d\Omega_y \\ &+ \epsilon_{mn}^* \oint_{\partial\Omega} \mathbb{C}_{pqk\ell} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y}) \Xi_{kmn}^T(\mathbf{n}_y)\mathbf{U}^{E,N}(1) n_\ell(\mathbf{y}) dS_y. \end{aligned} \tag{35}$$

Depending on whether $\mathbf{x}$ is inside or outside the inclusion, the domain integral in Eq. (35) has two different solutions. Denote the domain integrals

$$\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{I,\infty}(t) = -\int_{\Omega_e} \mathbb{C}_{pqmn} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y}) d\Omega_y, \quad \forall \mathbf{x} \in \Omega_e, \tag{36}$$

$$\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{E,\infty}(t) = -\int_{\Omega_e} \mathbb{C}_{pqmn} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y}) d\Omega_y, \quad \forall \mathbf{x} \in \Omega/\Omega_e, \tag{37}$$

and the boundary integral

$$\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{B,N}(t) = \oint_{\partial\Omega} \mathbb{C}_{pqk\ell} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y})\ \Xi_{kmn}^T(\mathbf{n_y})\mathbf{U}^{E,N}(1) n_\ell(\mathbf{y}) dS_y, \quad \forall \mathbf{x} \in \Omega. \tag{38}$$

Substituting Eqs. (36)–(38) into Eq. (35) and eliminating the circumference basis, one can find the following pair of algebraic equations for the radial coefficients:

$$\mathbf{U}^{I,N}(t) = \mathbf{U}^{I,\infty}(t) + \mathbf{U}^{B,N}(t), \quad 0 \le t \le a/H_0, \tag{39}$$

$$\mathbf{U}^{E,N}(t) = \mathbf{U}^{E,\infty}(t) + \mathbf{U}^{B,N}(t), \quad a/H_0 \le t \le 1. \tag{40}$$

Closed form expressions for $\mathbf{U}^{I,\infty}(t)$ and $\mathbf{U}^{E,\infty}(t)$ can be obtained by directly evaluating the domain integrals (36) and (37),

$$\begin{aligned} &\int_{\Omega_e} \mathbb{C}_{pqmn} G_{pi,q}^{\infty}(\mathbf{x}-\mathbf{y}) d\Omega_y \\ &= \frac{1}{8\pi(1-v)} \Bigg\{ (4v-3)\left(\int_{\Omega_e} \frac{\partial}{\partial x_j} \ln\ R\ d\Omega_y\right)\left(\frac{2v}{1-2v}\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) \\ &\quad + \frac{2v}{1-2v}\left[\int_{\Omega_e} \frac{\partial}{\partial x_j}\left(\frac{R_i R_j}{R^2}\right) d\Omega_y\right]\delta_{mn} + \left[\int_{\Omega_e} \frac{\partial}{\partial x_j}\left(\frac{R_i R_m}{R^2}\right) d\Omega_y\right]\delta_{jn} \\ &\quad + \left[\int_{\Omega_e} \frac{\partial}{\partial x_j}\left(\frac{R_i R_n}{R^2}\right) d\Omega_y\right]\delta_{jm} \Bigg\}, \quad \forall \mathbf{x} \in \Omega. \end{aligned} \tag{41}$$

For $\mathbf{x} \in \Omega_e$, the integration can easily be evaluated as

$$\Xi_{imn}^T(\mathbf{r})\mathbf{U}^{I,\infty}(t) = \frac{4v-1}{8(1-v)} x_i \delta_{mn} + \frac{3-4v}{8(1-v)}(x_m \delta_{in} + x_n \delta_{im}). \tag{42}$$

For $\mathbf{x} \in \Omega/\Omega_e$, the integrals involved in the above equation can be derived as

$$(a) \int_{\Omega_e} \frac{\partial}{\partial x_i}(\ln\ R) d\Omega_y = \frac{\pi a^2}{|\mathbf{x}|} r_i, \quad \forall \mathbf{x} \in \Omega/\Omega_e, \tag{43}$$

$$(b)\int_{\Omega_e}\frac{\partial}{\partial x_k}\left(\frac{R_iR_j}{R^2}\right)d\boldsymbol{\Omega}_y=\frac{\pi a^2}{|\mathbf{x}|}\left[\left(1-\frac{a^2}{2|\mathbf{x}|^2}\right)(r_i\delta_{jk}+r_j\delta_{ik})-\frac{a^2}{2|\mathbf{x}|^2}r_k\delta_{ij}\right.$$
$$\left.-2\left(1-\frac{a^2}{|\mathbf{x}|^2}\right)r_ir_jr_k\right],\quad \forall\mathbf{x}\in\boldsymbol{\Omega}/\boldsymbol{\Omega}_e. \tag{44}$$

Substituting Eqs. (43) and (44) into Eq. (41) yields

$$\Xi^T_{imn}(\mathbf{r})\mathbf{U}^{E,\infty}(t)=-\frac{1}{8(1-v)}\left\{\left[2(1-2v)\frac{a^2}{|\mathbf{x}|^2}-\frac{a^4}{|\mathbf{x}|^4}\right]x_i\delta_{mn}\right.$$
$$\left.+\left[2(2v-1)\frac{a^2}{|\mathbf{x}|^2}-\frac{a^4}{|\mathbf{x}|^4}\right](x_m\delta_{in}+x_n\delta_{im})-\left(4\frac{a^2}{|\mathbf{x}|^2}-\frac{a^4}{|\mathbf{x}|^4}\right)\frac{x_ix_mx_n}{|\mathbf{x}|^2}\right\}. \tag{45}$$

After reorganization, $\mathbf{U}^{I,\infty}(t)$ and $\mathbf{U}^{E,\infty}(t)$ are obtained as

$$\mathbf{U}^{I,\infty}(t)=\frac{H_0t}{8(1-v)}\begin{bmatrix}4v-1\\3-4v\\0\end{bmatrix}\quad\text{and}\quad \mathbf{U}^{E,\infty}(t)=\frac{\rho_0^2H_0}{8(1-v)\,t}\begin{bmatrix}4v-2+\frac{\rho_0^2}{t^2}\\2-4v+\frac{\rho_0^2}{t^2}\\4\left(1-\frac{\rho_0^2}{t^2}\right)\end{bmatrix}. \tag{46}$$

It is noted that, when subjected to the differential operator (33), the Eshelby tensor for an infinite RVE is recovered, i.e.,

$$\mathbf{S}^{I,\infty}(t)=\mathfrak{D}(t)\mathbf{U}^{I,\infty}(t)\quad\text{and}\quad \mathbf{S}^{E,\infty}(t)=\mathfrak{D}(t)\mathbf{U}^{E,\infty}(t)\ . \tag{47}$$

The boundary contribution $\mathbf{U}^{B,N}(t)$ can be evaluated directly from Eq. (38) as

$$\Xi^T_{imn}(\mathbf{r})\mathbf{U}^{B,N}(t)=\oint_{\partial\Omega}\mathscr{H}_{imn}(\mathbf{x},\mathbf{y})dS_y, \tag{48}$$

where

$$\mathscr{H}_{imn}=\mathbb{C}_{pqk\ell}G^{\infty}_{pi,q}(\mathbf{x}-\mathbf{y})\,\Xi^T_{kmn}(\mathbf{n_y})\mathbf{U}^{E,N}(1)n_\ell(\mathbf{y})$$
$$=\frac{1}{4\pi(1-v)R}\left\{U^{E,N}_1(1)\Big[(1-2v)(2n_in_p\ell_p\delta_{mn}-\ell_i\delta_{mn})+2\ell_in_p\ell_pn_q\ell_q\delta_{mn}\Big]\right.$$
$$+U^{E,N}_2(1)\Big[(1-2v)(n_mn_p\ell_p\delta_{in}+n_nn_p\ell_p\delta_{im}+n_in_m\ell_n+n_in_n\ell_m$$
$$-2n_mn_n\ell_i)+2n_m\ell_i\ell_nn_p\ell_p+2n_n\ell_i\ell_mn_p\ell_p\Big]$$
$$\left.+U^{E,N}_3(1)\Big[(1-2v)(2n_in_mn_n\ell_pn_p-n_mn_n\ell_i)+2n_mn_n\ell_in_p\ell_pn_q\ell_q\Big]\right\}.$$
$$\tag{49}$$

Integral (48) consists of the following independent parts, which are derived in the Appendix:

$$(i)\ \int_{\partial\Omega}\frac{1}{R}\ell_i dS_y = 0, \tag{50}$$

$$(ii)\ \int_{\partial\Omega}\frac{1}{R}n_i n_k \ell_k dS_y = \pi\ t\ r_i, \tag{51}$$

$$(iii)\ \int_{\partial\Omega}\frac{1}{R}n_i n_j \ell_k dS_y = \frac{\pi}{2}t(r_i\delta_{jk} + r_j\delta_{ik} - r_k\delta_{ij}), \tag{52}$$

$$(iv)\ \int_{\partial\Omega}\frac{1}{R}n_i \ell_j \ell_k n_p \ell_p dS_y = \frac{\pi}{4}t(3\ r_i\delta_{jk} - r_j\delta_{ik} - r_k\delta_{ij}), \tag{53}$$

$$(v)\ \int_{\partial\Omega}\frac{1}{R}n_i n_j n_k n_p \ell_p dS_y = \frac{\pi}{4}t(1-t^2)(r_i\delta_{jk} + r_j\delta_{ik} + r_k\delta_{ij}) + \pi\ t^3\ r_i r_j r_k, \tag{54}$$

$$(vi)\ \int_{\partial\Omega}\frac{1}{R}\ell_i n_p \ell_p n_q \ell_q dS_y = 0, \tag{55}$$

$$(vii)\ \int_{\partial\Omega}\frac{1}{R}n_i n_j \ell_k n_p \ell_p n_q \ell_q dS_y$$

$$= \frac{\pi}{2}t\big(1-t^2\big)(r_i\delta_{jk} + r_j\delta_{ik}) - \frac{\pi}{2}t\left(1 - \frac{t^2}{2}\right)r_k\delta_{ij} + \frac{\pi}{2}\ t^3\ r_i r_j r_k. \tag{56}$$

Integral (48) can be integrated exactly after substitution of Eqs. (50)–(56). After some algebraic manipulation, the final result is expressed in compact matrix form as

$$\mathbf{U}^{B,N}(t) = \mathbf{K}(t)\ \mathbf{U}^{E,N}(1), \tag{57}$$

where

$$\mathbf{K}(t) = \frac{1}{4(1-v)}\begin{bmatrix} 2(1-2v)t & (1-4v)t & -2vt + vt^3 \\ 0 & t & t + \frac{2v-3}{2}t^3 \\ 0 & 0 & (3-4v)t^3 \end{bmatrix}. \tag{58}$$

The term $\mathbf{U}^{B,N}(t)$ represents the Neumann boundary correction of the disturbance displacement field due to a finite RVE. This reduces the system of Eqs. (39) and (40) to

$$\mathbf{U}^{I,N}(t) = \mathbf{U}^{I,\infty}(t) + \mathbf{K}(t)\ \mathbf{U}^{E,N}(1), \quad 0 \le t \le a/H_0, \tag{59}$$

$$\mathbf{U}^{E,N}(t) = \mathbf{U}^{E,\infty}(t) + \mathbf{K}(t)\ \mathbf{U}^{E,N}(1), \quad a/H_0 \le t < 1. \tag{60}$$

Assuming that $\mathbf{U}^{I,N}(t)$ and $\mathbf{U}^{E,N}(t)$ depend on t continuously we let $t \to 1$. One can solve for $\mathbf{U}^{E,N}(1)$ by letting $t = 1$ in Eq. (60), i.e.,

$$\mathbf{U}^{E,N}(1) = \Big[1 - \mathbf{K}(1)\Big]^{-1}\mathbf{U}^{E,\infty}(1), \tag{61}$$

which gives

$$\mathbf{U}^{E,N}(1) = H_0 \begin{bmatrix} \frac{1}{2}\rho_0^2(\rho_0^2 - 1) \\ \frac{1}{2}\rho_0^4 \\ 2\rho_0^2(1-\rho_0^2) \end{bmatrix}. \tag{62}$$

Substituting Eq. (62) back into Eqs. (59) and (60), one can first find

$$\mathbf{U}^{B,N}(t) = \frac{\rho_0^2 H_0}{8(1-v)} \begin{bmatrix} (3\rho_0^2 - 4v - 2)\, t + 4v(1-\rho_0^2)t^3 \\ (4-3\rho_0^2)\, t + 2(2v-3)(1-\rho_0^2)t^3 \\ 4(3-4v)(1-\rho_0^2)t^3 \end{bmatrix}, \tag{63}$$

and subsequently one can solve for both $\mathbf{U}^{I,N}(t)$ and $\mathbf{U}^{E,N}(t)$, which are the radial basis for the interior and exterior disturbance displacement fields in a finite domain,

$$\mathbf{U}^{I,N}(t) = \frac{H_0}{8(1-v)} \begin{bmatrix} 4v-1 \\ 3-4v \\ 0 \end{bmatrix} + \frac{\rho_0^2 H_0}{8(1-v)} \begin{bmatrix} (3\rho_0^2 - 4v - 2)\, t + 4v(1-\rho_0^2)t^3 \\ (4-3\rho_0^2)\, t + 2(2v-3)(1-\rho_0^2)t^3 \\ 4(3-4v)(1-\rho_0^2)t^3 \end{bmatrix}, \quad 0 \le t \le a/H_0, \tag{64}$$

$$\mathbf{U}^{E,N}(t) = \frac{\rho_0^2 H_0}{8(1-v)} \begin{bmatrix} \dfrac{4v-2}{t} + \dfrac{\rho_0^2}{t^3} \\ \dfrac{2-4v}{t} + \dfrac{\rho_0^2}{t^3} \\ \dfrac{4}{t} - 4\dfrac{\rho_0^2}{t^3} \end{bmatrix} + \frac{\rho_0^2 H_0}{8(1-v)} \begin{bmatrix} (3\rho_0^2 - 4v - 2)\, t + 4v(1-\rho_0^2)t^3 \\ (4-3\rho_0^2)\, t + 2(2v-3)(1-\rho_0^2)t^3 \\ 4(3-4v)(1-\rho_0^2)t^3 \end{bmatrix}, \quad a/H_0 \le t \le 1. \tag{65}$$

The radial coefficients for the Eshelby tensor can be obtained by applying the differentiation operator (33) to Eqs. (64) and (65),

$$\mathbf{S}^{I,N}(t) = \mathfrak{D}(t)\mathbf{U}^{I,N}(t) = \frac{1}{8(1-v)} \begin{bmatrix} 4v-1 \\ 3-4v \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{\rho_0^2}{8(1-v)} \begin{bmatrix} -2(1+2v) + 3\rho_0^2 + 12(1-\rho_0^2)v\, t^2 \\ 4 - 3\rho_0^2 - 6(1-\rho_0^2)\, t^2 \\ 12(1-2v)(1-\rho_0^2)\, t^2 \\ 0 \\ 0 \end{bmatrix}, \tag{66}$$

$$\mathbf{S}^{E,N}(t) = \mathfrak{D}(t)\mathbf{U}^{E,N}(t)$$

$$= \frac{\rho_0^2}{8(1-v)} \begin{bmatrix} \dfrac{-2(1+2v)}{t^2} + \dfrac{9\rho_0^2}{t^4} \\ \dfrac{2}{t^2} - \dfrac{3\rho_0^2}{t^4} \\ \dfrac{4(1+2v)}{t^2} - \dfrac{12\rho_0^2}{t^4} \\ \dfrac{4}{t^2} - \dfrac{12\rho_0^2}{t^4} \\ -\dfrac{16}{t^2} + \dfrac{24\rho_0^2}{t^4} \end{bmatrix} + \frac{\rho_0^2}{8(1-v)} \begin{bmatrix} -2(1+2v) + 3\rho_0^2 + 12(1-\rho_0^2)v\,t^2 \\ 4 - 3\rho_0^2 - 6(1-\rho_0^2)\,t^2 \\ 12(1-2v)(1-\rho_0^2)\,t^2 \\ 0 \\ 0 \end{bmatrix}. \tag{67}$$

The exact expressions for the Neumann-Eshelby tensors for a circular inclusion embedded in a circular RVE under prescribed traction boundary condition are then given as follows:

$$\begin{aligned} \mathbb{S}^{I,N}_{ijmn}(\mathbf{x}) = \frac{1}{8(1-v)} \Big\{ & \Big[(4v-1)(1-\rho_0^2) - 3\rho_0^2(1-\rho_0^2)(1-4v\,t^2)\Big]\delta_{ij}\delta_{mn} \\ & + \Big[(3-4v) + \rho_0^2 + 3\rho_0^2(1-\rho_0^2)(1-2\,t^2)\Big](\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\ & + \Big[12(1-2v)\rho_0^2(1-\rho_0^2)\,t^2\Big]\delta_{ij}r_m r_n \Big\}, \quad \forall \mathbf{x} \in \Omega_e, \end{aligned} \tag{68}$$

$$\begin{aligned} \mathbb{S}^{E,N}_{ijmn}(\mathbf{x}) = \frac{\rho_0^2}{8(1-v)} \Big\{ & \Big[-2(1+2v)\Big(\frac{1}{t^2}+1\Big) + 12vt^2 + \rho_0^2\Big(\frac{9}{t^4} + 3 - 12vt^2\Big)\Big]\delta_{ij}\delta_{mn} \\ & + \Big[\frac{2}{t^2} + 4 - 6t^2 - 3\rho_0^2\Big(\frac{1}{t^4} + 1 - 2t^2\Big)\Big](\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\ & + \Big[4\Big(\frac{1+2v}{t^2} + 3(1-2v)\,t^2\Big) - 12\rho_0^2\Big(\frac{1}{t^4} + (1-2v)t^2\Big)\Big]\delta_{ij}r_m r_n \\ & + \Big[\frac{4}{t^2}\Big(1 - \frac{3\rho_0^2}{t^2}\Big)\Big]\delta_{mn}r_i r_j \\ & + \Big[\frac{8}{t^2}\Big(\frac{3\rho_0^2}{t^2} - 2\Big)\Big]r_i r_j r_m r_n \Big\}, \quad \forall \mathbf{x} \in \Omega/\Omega_e. \end{aligned} \tag{69}$$

Expressions (64), (65), (68), and (69) can be re-cast into a unified form,

$$\mathbf{S}^{\bullet,N}(t) = \mathbf{S}^{\bullet,\infty}(t) + \mathbf{S}^{B,N}(t), \quad 0 \le t \le 1, \tag{70}$$

$$\mathbf{U}^{\bullet,N}(t) = \mathbf{U}^{\bullet,\infty}(t) + \mathbf{U}^{B,N}(t), \quad 0 \le t \le 1. \tag{71}$$

We emphasize that the solution in a finite RVE is essentially composed of the infinite domain solution and the boundary correction. The components of each term are plotted in Fig. 2 and Fig. 3a, c and e for an inclusion size $\rho_0 = 0.2$ and Poisson's ratio $v = 0.3$. Note that all Eshelby

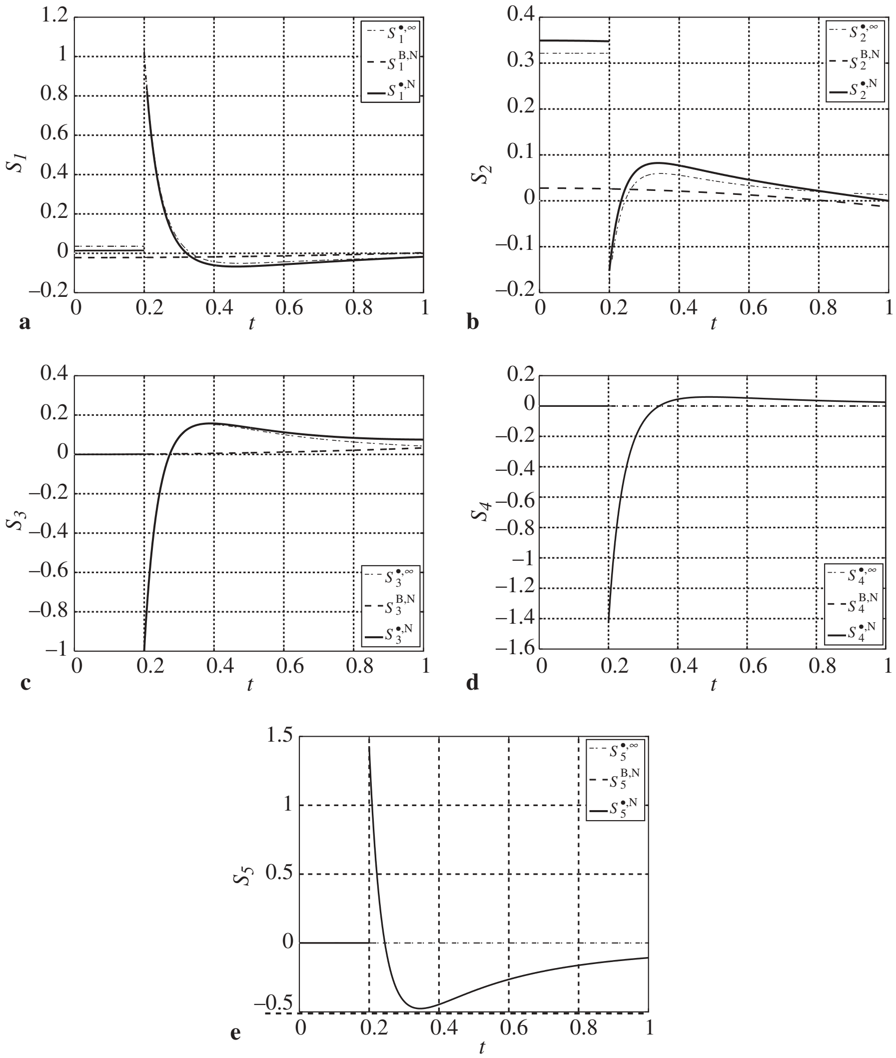

Fig. 2. Coeffcients of the Neumann-Eshelby tensor $S_1(t), S_2(t), S_3(t), S_4(t)$ and $S_5(t)$

tensor coefficients have a jump at the inclusion/matrix interface $t = \rho_0$, while the disturbance displacement field remains continuous within the RVE.

4 Disturbance traction field

The induced disturbance traction field on a set of successive concentric circular surfaces can be determined using

$$t_i^d(\mathbf{x}) = \sigma_{ji}^d(\mathbf{x})\ r_j(\mathbf{x}). \tag{72}$$

In view of (11) and the Eshelby tensors (68)–(69), the disturbance traction field can be specified as

$$t_i^d(\mathbf{x}) = \begin{cases} r_j(\mathbf{x})\ \mathbb{C}_{ijk\ell}\Big(\mathbb{S}^{I,N}_{k\ell mn}(\mathbf{x}) - \mathbb{I}^{(4s)}_{k\ell mn}\Big)\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega_e, \\ r_j(\mathbf{x})\ \mathbb{C}_{ijk\ell}\ \mathbb{S}^{E,N}_{k\ell mn}(\mathbf{x})\ \epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega/\Omega_e. \end{cases} \tag{73}$$

where $\mathbb{I}^{(4s)}_{k\ell mn}$ is the fourth-order symmetric identity tensor. It is noted that it also falls into our new category of a *fourth-order radial isotropic tensor*.

Simple calculation reveals that $t_i^d(\mathbf{x})$ can also be characterized using a third-order radial isotropic tensor $\mathbb{T}_{imn}(\mathbf{x})$, which can be further decomposed into a radial and circumference basis such that

$$t_i^d(\mathbf{x}) = \begin{cases} \mathbb{T}^{I,N}_{imn}(\mathbf{r})\ \epsilon^*_{mn} = \ \Xi^T_{imn}(\mathbf{r})\mathbf{T}^{I,N}(t)\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega_e, \\ \mathbb{T}^{E,N}_{imn}(\mathbf{r})\epsilon^*_{mn} = \ \Xi^T_{imn}(\mathbf{n})\mathbf{T}^{E,N}(t)\epsilon^*_{mn}, & \forall \mathbf{x} \in \Omega/\Omega_e, \end{cases} \tag{74}$$

with the radial basis being determined by

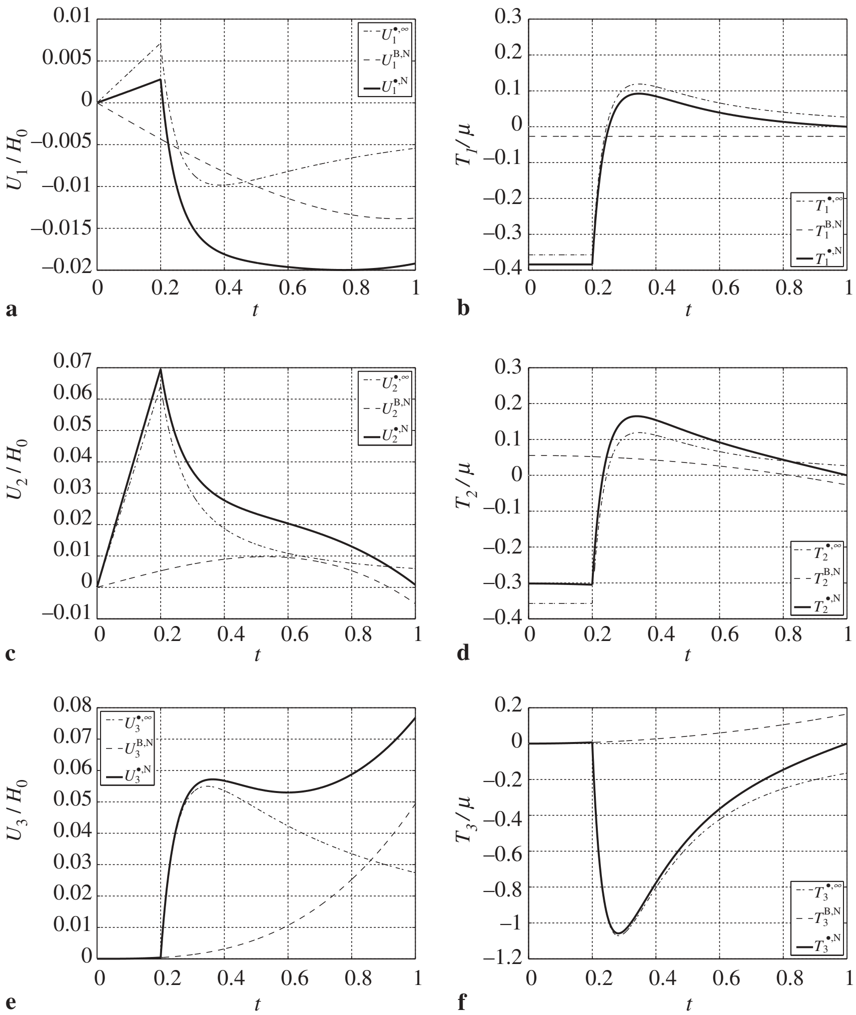

Fig. 3. Coefficients of the Disturbance Displacement Field $U_i(t)$ $i = 1,2,3$ ($\mathbf{a},\mathbf{c},\mathbf{e}$); Coefficients of the Disturbance Traction Field $T_i(t)$ $i = 1,2,3$ ($\mathbf{b},\mathbf{d},\mathbf{f}$)

$$\mathbf{T}^{I,N}(t) = \frac{2\mu}{1-2\nu}\begin{bmatrix} S_1^{I,N}(t) + 2\nu S_2^{I,N}(t) + (1-\nu)S_4^{I,N}(t) - \nu \\ (1-2\nu)S_2^{I,N}(t) - (1-2\nu)/2 \\ S_3^{I,N}(t) + (1-\nu)S_5^{I,N}(t) \end{bmatrix}, \quad 0 \le t \le a/H_0, \tag{75}$$

$$\mathbf{T}^{E,N}(t) = \frac{2\mu}{1-2\nu}\begin{bmatrix} S_1^{E,N}(t) + 2\nu S_2^{E,N}(t) + (1-\nu)\, S_4^{E,N}(t) \\ (1-2\nu)S_2^{E,N}(t) \\ S_3^{E,N}(t) + (1-\nu)S_5^{E,N}(t) \end{bmatrix}, \quad a/H_0 \le t \le 1. \tag{76}$$

It is readily verified that when $t = 1$,

$$\mathbf{T}^{E,N}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{77}$$

This means that the traction vector vanishes at the boundary of the RVE, i.e., $t_i^d(\mathbf{x}) \equiv 0$ for $\forall \mathbf{x} \in \partial\Omega$. This verifies that the prescribed Neumann boundary condition is exactly satisfied by the obtained Neumann-Eshelby solution.

Figure 3b, d and f show profiles of the disturbance traction coefficients of $\mathbf{T}^{\bullet,\infty}(t)$, $\mathbf{T}^{B,N}(t)$ and $\mathbf{T}^{\bullet,N}(t)$, where we chose $\nu = 0.3$ and $\rho_0 = 0.2$. Note that all coefficients of the disturbance displacement and traction fields are continuous and that $\mathbf{T}^{E,N}(t) = \mathbf{0}$ at $t = 1$.

5 Applications

With the Neumann-Eshelby tensor derived, homogenization theory needs to be revisited to take into account the boundary effect of the RVE and the size effect of the inclusion.

Consider the averages

$$\langle t^2 \rangle_{\Omega_e} = \frac{1}{2}\rho_0^2 \text{ and } \langle t^2 n_m n_n \rangle_{\Omega_e} = \frac{1}{4}\rho_0^2\, \delta_{mn}, \tag{78}$$

and the volume fraction of the inclusion in an RVE

$$f := \rho_0^2 = \frac{a^2}{H_0^2}. \tag{79}$$

The average interior Neumann-Eshelby tensor from Eq. (68) has the following form:

$$\langle \mathbb{S}_{ijmn}^{I,N} \rangle_{\Omega_e} = s_1\, \mathbb{E}_{ijmn}^{(1)} + s_2\, \mathbb{E}_{ijmn}^{(2)}, \tag{80}$$

with

$$s_1 = \frac{1 + (1-2\nu)f}{2(1-\nu)}, \quad s_2 = \frac{(3-4\nu) + f(4 - f(6-3f))}{4(1-\nu)}, \tag{81}$$

and

$$\mathbb{E}^{(1)}_{ijmn} = \frac{1}{2}\delta_{ij}\delta_{mn}, \tag{82}$$

$$\mathbb{E}^{(2)}_{ijmn} = \frac{1}{2}\Big(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn}\Big). \tag{83}$$

One may compare this result with the interior Eshelby tensor of an infinite space,

$$\mathbb{S}^{I,\infty}_{ijmn} = s^0_1\ \mathbb{E}^{(1)}_{ijmn} + s^0_2\ \mathbb{E}^{(2)}_{ijmn} = \frac{1}{2(1-\nu)}\mathbb{E}^{(1)}_{ijmn} + \frac{3-4\nu}{4(1-\nu)}\mathbb{E}^{(2)}_{ijmn}. \tag{84}$$

Note the average interior Neumann-Eshelby tensor is different from the average interior Dirichlet-Eshelby tensor of part I.

Note the following limiting cases:

(i) when $f = 0$, $s_1 = s^0_1$, and $s_2 = s^0_2$;
(ii) when $f = 1$, $s_1 = 1$, and $s_2 = 1$.

The newly derived Neumann-Eshelby tensor $\langle \mathbb{S}^{I,\infty}_{ijmn}\rangle_{\Omega_e}$ is used in the homogenization of a single inclusion problem. For comparison, we consider the same material properties as used in Part I, where the ratios of inclusion and matrix properties are: bulk modulus $K_e/K = 10$, shear modulus $\mu_e/\mu = 5$, and Poisson's ratio $\nu_e/\nu = 3$. Effective material properties K_{eff}, μ_{eff} and ν_{eff} are evaluated using the dilute homogenization method (e.g., Nemat-Nasser and Hori [9]) *under prescribed traction*. The homogenization results are depicted in Fig. 4 in comparison with those obtained using $\mathbb{S}^{I,\infty}_{ijmn}$. It is well known that the dilute homogenization gives poor predictions for effective material properties when the volume fraction of the inclusion is large (dashed lines). By using the new Neumann-Eshelby tensor, satisfactory results have been obtained for the whole range of the volume fraction of the inclusion ($f \in [0,1]$).

To understand the nature of the Neumann-Eshelby tensor, we calculate the sums of both the interior and the exterior Neumann-Eshelby tensors, $\mathbb{S}^{I,N}_{iijj}(\mathbf{x})$ and $\mathbb{S}^{E,N}_{iijj}(\mathbf{x})$,

$$\begin{aligned}\mathbb{S}^{I,N}_{iijj}(\mathbf{x}) &= \mathbb{S}^{I,N}_{1111}(\mathbf{x}) + \mathbb{S}^{I,N}_{2222}(\mathbf{x}) + \mathbb{S}^{I,N}_{1122}(\mathbf{x}) + \mathbb{S}^{I,N}_{2211}(\mathbf{x})\\ &= 4S^{I,N}_1(t) + 4S^{I,N}_2(t) + 2S^{I,N}_3(t) + 2S^{I,N}_4(t) + S^{I,N}_5(t)\\ &= \frac{1+(1-2\nu)f}{1-\nu} = 2s_1\end{aligned} \tag{85}$$

and

$$\begin{aligned}\mathbb{S}^{E,N}_{iijj}(\mathbf{x}) &= \mathbb{S}^{E,N}_{1111}(\mathbf{x}) + \mathbb{S}^{E,N}_{2222}(\mathbf{x}) + \mathbb{S}^{E,N}_{1122}(\mathbf{x}) + \mathbb{S}^{E,N}_{2211}(\mathbf{x})\\ &= 4S^{E,N}_1(t) + 4S^{E,N}_2(t) + 2S^{E,N}_3(t) + 2S^{E,N}_4(t) + S^{E,N}_5(t)\\ &= \frac{(1-2\nu)f}{1-\nu}.\end{aligned} \tag{86}$$

Again, the dependence of both $\mathbb{S}^{I,N}_{ijmn}(\mathbf{x})$ and $\mathbb{S}^{E,N}_{ijmn}(\mathbf{x})$ on the position vector is cancelled out under the prescribed uniform biaxial eigenstrain distribution, but they still have explicit dependence on the volume fraction of the inclusion.

Compare the Neumann-Eshelby tensor with the Dirichlet-Eshelby tensor obtained in part I: since $(1-2\nu) \geq 0$,

$$\frac{1+(1-2\nu)f}{1-\nu} > \frac{1-f}{1-\nu} \ \Rightarrow\ \mathbb{S}^{I,N}_{iijj} > \mathbb{S}^{I,D}_{iijj}, \tag{87}$$

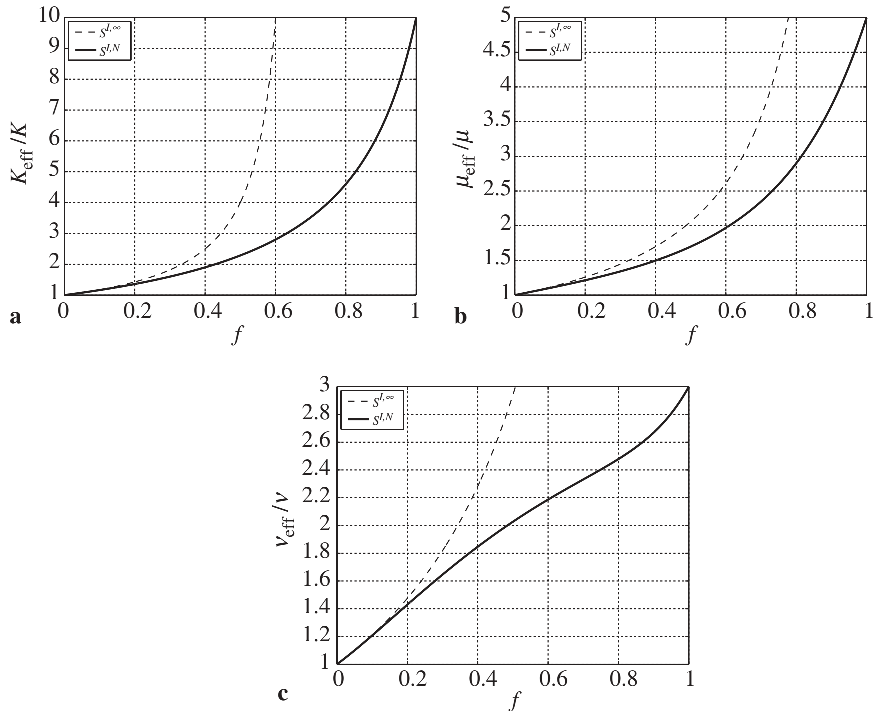

Fig. 4. Comparisons of homogenization results between the methods using $S^{I,\infty}_{ijmn}$ and the methods using $S^{I,N}_{ijmn}$

$$\frac{(1-2\nu)f}{1-\nu} > \frac{-f}{1-\nu} \;\Rightarrow\; \mathbb{S}^{E,N}_{iijj} > \mathbb{S}^{E,D}_{iijj}. \tag{88}$$

This indicates that the dilatational sum of the Neumann-Eshelby tensor is always larger than that of the Dirichlet-Eshelby tensor, which means that under the same prescribed bi-axial eigenstrain field the disturbance strain field corresponding to a prescribed traction boundary condition will be larger than the disturbance strain field corresponding to a prescribed displacement boundary condition.

It may be interesting to note that

$$\mathbb{S}^{I,N}_{iijj} - \mathbb{S}^{E,N}_{iijj} = \mathbb{S}^{I,\infty}_{iijj} = \frac{1}{1-\nu} = 2\, s^0_1. \tag{89}$$

6 Conclusions

In this paper, we solved the dual problem to part I: that is the elastic fields of a 2D plane strain circular inclusion in a finite representative volume element under the prescribed traction boundary condition. By doing so, we have found the so-called Neumann-Eshelby tensors in both the interior and exterior region of a circular RVE with a circular inclusion.

As in the case of the Dirichlet-Eshelby tensors, it has been shown that the Neumann-Eshelby tensors do not depend on the prescribed boundary data, and they only depend on the volume fraction of the inclusion and the position where the tensors are being evaluated.

Applications of the Neumann-Eshelby tensor shows remarkable improvement in a simple homogenization procedure. We have obtained the three-dimensional counterparts of both the Dirichlet-Eshelby tensor and the Neumann-Eshelby tensor, and have applied these results to develop new variational bounds. They will be reported in later papers.

Appendix

Integration formulas

In this Appendix, we document the detailed integration procedures in evaluating the seven integrations listed in Eqs. (50)–(56).

Define:

$$t =: \frac{|\mathbf{x}|}{H_0}, \tag{90}$$

$$R =: |\mathbf{y} - \mathbf{x}| = H_0\sqrt{1 - 2t\cos\phi + t^2}, \tag{91}$$

and

$$\ell = \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} = \frac{1}{\sqrt{1 - 2t\cos\phi + t^2}}\begin{bmatrix}\cos(\varphi + \phi) - t\cos\varphi \\ \sin(\varphi + \phi) - t\sin\varphi\end{bmatrix}. \tag{92}$$

According to Fig. 1, we define the unit normal vectors at $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \partial\Omega$:

$$\mathbf{n}(\mathbf{y}) = \frac{\mathbf{y}}{|\mathbf{y}|} = \begin{bmatrix}\cos(\varphi + \phi) \\ \sin(\varphi + \phi)\end{bmatrix}, \quad \text{and } \mathbf{y} = H_0\mathbf{n}(\mathbf{y}), \tag{93}$$

$$\mathbf{r}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} = \begin{bmatrix}\cos\varphi \\ \sin\varphi\end{bmatrix}, \quad \text{and } \mathbf{x} = tH_0\mathbf{r}(\mathbf{x}). \tag{94}$$

Considering $\mathbf{x} + R\ell = \mathbf{y}$, we have

$$\ell_i = \frac{H_0}{R}(n_i(\mathbf{y}) - \bar{x}_i), \tag{95}$$

or

$$n_i(\mathbf{y}) = \frac{R}{H_0}\ell_i + \bar{x}_i, \tag{96}$$

where

$$\bar{x}_i =: \frac{x_i}{H_0} = t\ r_i(\mathbf{x}) \tag{97}$$

Equation (96) is useful in the following derivations to substitute $\mathbf{n}$ in the integrands for $\boldsymbol{\ell}$. Another trick usually applied is to substitute unity into the integrand, which reads

$$\begin{aligned}\ell_s\ell_s &= \left(\frac{H_0}{R}\right)^2(n_s - \bar{x}_s)^2 = \left(\frac{H_0}{R}\right)^2(1 + t^2 - 2n_s\bar{x}_s) = \left(\frac{H_0}{R}\right)^2\left(1 + t^2 - 2\left(\frac{R}{H_0}\ell_s + \bar{x}_s\right)\bar{x}_s\right) \\ &= \left(\frac{H_0}{R}\right)^2(1 - t^2) - 2\left(\frac{H_0}{R}\right)\ell_s\bar{x}_s \equiv 1,\end{aligned} \tag{98}$$

Elemental integrals

To facilitate the integrations, we shall break the integrals (50)–(56) into the following elemental integrals, which have been proved in the Appendix of Part I.

The seven elemental integrals are

$$\text{(i)}\quad \int_0^{2\pi} d\phi = 2\pi, \tag{99}$$

$$\text{(ii)}\quad \int_0^{2\pi} \frac{1}{R}\ell_i d\phi = 0, \tag{100}$$

$$\text{(iii)}\quad \int_0^{2\pi} \ell_i\ell_j d\phi = \pi\delta_{ij}, \tag{101}$$

$$\text{(iv)}\quad \int_0^{2\pi} \frac{1}{R}\ell_i\ell_j\ell_m d\phi = 0, \tag{102}$$

$$\text{(v)}\quad \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_n d\phi = \frac{\pi}{4}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{103}$$

$$\text{(vi)}\quad \int_0^{2\pi} \frac{1}{R}\ell_i\ell_j\ell_m\ell_n\ell_r d\phi = 0, \tag{104}$$

$$\begin{aligned}\text{(vii)}\quad \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_n\ell_r\ell_s d\phi = \frac{\pi}{24}\Big(&\delta_{ij}\delta_{mn}\delta_{rs} + \delta_{im}\delta_{jn}\delta_{rs} + \delta_{in}\delta_{jm}\delta_{rs} + \delta_{ir}\delta_{mn}\delta_{js} + \delta_{is}\delta_{mn}\delta_{jr}\\ &+ \delta_{ij}\delta_{mr}\delta_{ns} + \delta_{im}\delta_{jr}\delta_{ns} + \delta_{in}\delta_{jr}\delta_{ms} + \delta_{ir}\delta_{mj}\delta_{ns} + \delta_{is}\delta_{mj}\delta_{nr}\\ &+ \delta_{ij}\delta_{ms}\delta_{nr} + \delta_{im}\delta_{js}\delta_{nr} + \delta_{in}\delta_{js}\delta_{mr} + \delta_{ir}\delta_{ms}\delta_{nj} + \delta_{is}\delta_{mr}\delta_{nj}\Big).\end{aligned} \tag{105}$$

Utilizing Eq. (98), one can further find the following identities:

$$\int_0^{2\pi} \frac{R}{H_0}\ \ell_i d\phi = -2\bar{x}_s \int_0^{2\pi} \ell_i\ell_s d\phi,$$

$$\int_0^{2\pi} \frac{R}{H_0}\ \ell_i\ell_j\ell_m d\phi = -2\bar{x}_s \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_s d\phi, \tag{106}$$

$$\int_0^{2\pi} \frac{R}{H_0}\ \ell_i\ell_j\ell_m\ell_n\ell_r d\phi = -2\bar{x}_s \int_0^{2\pi} \ell_i\ell_j\ell_m\ell_n\ell_r\ell_s d\phi. \tag{107}$$

The seven integrals

(i) Considering integration (100) it is trivial to show that

$$I_I = \int_{\partial\Omega} \frac{1}{R}\ell_i dS_y = 0\ . \tag{108}$$

(ii) Likewise, by using the elemental integrals, we can show

$$\begin{aligned} I_{II} &= \int_{\partial\Omega} \frac{1}{R} n_i n_k \ell_k dS_y = \int_0^{2\pi} \frac{H_0}{R} n_i n_k \ell_k d\phi\\ &= \int_0^{2\pi} \frac{R}{H_0}\ell_i d\phi + \bar{x}_i \int_0^{2\pi} d\phi + \bar{x}_k \int_0^{2\pi} \ell_i\ell_k d\phi + \bar{x}_i\bar{x}_k \int_0^{2\pi} \frac{H_0}{R}\ell_k d\phi\\ &= \pi tr_i. \end{aligned} \tag{109}$$

(iii) It can also be readily shown that

$$I_{III} = \int_{\partial\Omega} \frac{1}{R} n_i n_j \ell_k dS_y = \int_0^{2\pi} \frac{H_0}{R} n_i n_j \ell_k d\phi$$

$$= \int_0^{2\pi} \frac{R}{H_0} \ell_i \ell_j \ell_k d\phi + \bar{x}_j \int_0^{2\pi} \ell_i \ell_k d\phi + \bar{x}_i \int_0^{2\pi} \ell_j \ell_k d\phi + \bar{x}_i \bar{x}_j \int_0^{2\pi} \frac{H_0}{R} \ell_k d\phi$$

$$= \frac{\pi}{2} t(\delta_{ik} r_j + \delta_{jk} r_i - \delta_{ij} r_k). \tag{110}$$

(iv) Integral I_{IV} follows as:

$$I_{IV} = \int_{\partial\Omega} \frac{1}{R} n_i \ell_j \ell_k n_p \ell_p dS_y = \int_0^{2\pi} \frac{H_0}{R} n_i \ell_j \ell_k n_p \ell_p d\phi$$

$$= \int_0^{2\pi} \frac{R}{H_0} \ell_i \ell_j \ell_k d\phi + \bar{x}_i \int_0^{2\pi} \ell_j \ell_k d\phi + \bar{x}_p \int_0^{2\pi} \ell_i \ell_j \ell_k \ell_p d\phi + \bar{x}_i \bar{x}_p \int_0^{2\pi} \frac{H_0}{R} \ell_i \ell_k \ell_p d\phi$$

$$= \frac{\pi}{4} t(3\ \delta_{jk} r_i - \delta_{ik} r_j - \delta_{ij} r_k). \tag{111}$$

(v) By substituting Eq. (96), integral I_V is written as:

$$I_V = \int_{\partial\Omega} \frac{1}{R} n_i n_j n_k n_\ell \ell_\ell dS_y$$

$$= \int_0^{2\pi} \frac{H_0}{R} \left(\frac{R}{H_0} \ell_i + \bar{x}_i \right) \left(\frac{R}{H_0} \ell_j + \bar{x}_j \right) \left(\frac{R}{H_0} \ell_k + \bar{x}_k \right) \left(\frac{R}{H_0} \ell_\ell + \bar{x}_\ell \right) \ell_\ell d\phi$$

$$= \int_0^{2\pi} \Bigg[\left(\frac{R}{H_0} \right)^3 \ell_i \ell_j \ell_k + \left(\frac{R}{H_0} \right)^2 \ell_i \ell_j \ell_k \ell_\ell \bar{x}_\ell + \left(\frac{R}{H_0} \right)^2 (\ell_i \ell_j \bar{x}_k + \ell_i \ell_k \bar{x}_j + \ell_j \ell_k \bar{x}_i)$$

$$+ \left(\frac{R}{H_0} \right) (\ell_i \ell_j \ell_\ell \bar{x}_\ell \bar{x}_k + \ell_i \ell_k \ell_\ell \bar{x}_\ell \bar{x}_j + \ell_j \ell_k \ell_\ell \bar{x}_\ell \bar{x}_i) + \left(\frac{R}{H_0} \right) (\ell_i \bar{x}_j \bar{x}_k + \ell_k \bar{x}_i \bar{x}_j + \ell_j \bar{x}_i \bar{x}_k)$$

$$+ \ell_k \ell_\ell \bar{x}_i \bar{x}_j \bar{x}_\ell + \ell_i \ell_\ell \bar{x}_j \bar{x}_k \bar{x}_\ell + \ell_j \ell_\ell \bar{x}_i \bar{x}_k \bar{x}_\ell + \bar{x}_i \bar{x}_j \bar{x}_k + \left(\frac{H_0}{R} \right) \bar{x}_i \bar{x}_j \bar{x}_k \bar{x}_\ell \ell_\ell \Bigg] d\phi. \tag{112}$$

Three types of integrals are involved in the above besides the seven basic integrals:

$$\int_0^{2\pi} \left(\frac{R}{H_0} \right)^2 \ell_i \ell_j \ell_k \ell_\ell d\phi = (1 - t^2) \int_0^{2\pi} \ell_i \ell_j \ell_k \ell_\ell d\phi - 2\bar{x}_s \int_0^{2\pi} \left(\frac{R}{H_0} \right) \ell_i \ell_j \ell_k \ell_\ell \ell_s d\phi$$

$$= \frac{\pi}{4} (1 - t^2)(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell}) + 4\bar{x}_r \bar{x}_s \int_0^{2\pi} \ell_i \ell_j \ell_k \ell_\ell \ell_r \ell_s d\phi$$

$$= \frac{\pi}{4} (1 - \frac{1}{3} t^2)(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell})$$

$$+ \frac{\pi}{3} (\delta_{ij}\bar{x}_k \bar{x}_\ell + \delta_{k\ell}\bar{x}_i \bar{x}_j + \delta_{ik}\bar{x}_j \bar{x}_\ell + \delta_{i\ell}\bar{x}_j \bar{x}_k + \delta_{kj}\bar{x}_i \bar{x}_\ell + \delta_{j\ell}\bar{x}_i \bar{x}_k), \tag{113}$$

$$\int_0^{2\pi}\left(\frac{R}{H_0}\right)^3 \ell_i\ell_j\ell_k d\phi = \int_0^{2\pi}\left(\frac{R}{H_0}\right)^3 \ell_i\ell_j\ell_k\ell_s\ell_s d\phi$$
$$= \left(1-t^2\right)\int_0^{2\pi}\left(\frac{R}{H_0}\right)\ell_i\ell_j\ell_k d\phi - 2\bar{x}_s\int_0^{2\pi}\left(\frac{R}{H_0}\right)^2 \ell_i\ell_j\ell_k\ell_s d\phi$$
$$= -\pi(\delta_{ij}\bar{x}_k + \delta_{ik}\bar{x}_j + \delta_{jk}\bar{x}_i) - 2\pi\bar{x}_i\bar{x}_j\bar{x}_k, \tag{114}$$

$$\int_0^{2\pi}\left(\frac{R}{H_0}\right)^2 \ell_i\ell_j d\phi = \int_0^{2\pi}\left(\frac{R}{H_0}\right)^2 \ell_i\ell_j\ell_s\ell_s d\phi$$
$$= \left(1-t^2\right)\int_0^{2\pi}\ell_i\ell_j d\phi - 2\bar{x}_s\int_0^{2\pi}\left(\frac{R}{H_0}\right)\ell_i\ell_j\ell_s d\phi$$
$$= \left(1-t^2\right)\pi\delta_{ij} + 4\bar{x}_r\bar{x}_s\int_0^{2\pi}\ell_i\ell_j\ell_r\ell_s d\phi$$
$$= \pi(\delta_{ij} + 2\bar{x}_i\bar{x}_j). \tag{115}$$

Substituting all these integrals into Eq. (112), after lengthy algebra, it can be avaluated as

$$I_V = \frac{\pi}{4}t\left(1-t^2\right)(\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i) + \pi t^3 r_i r_j r_k. \tag{116}$$

(6) Use Eq. (96) to simplify

$$n_p\ell_p = \left(\frac{R}{H_0}\ell_p + \bar{x}_p\right)\ell_p = \frac{R}{H_0} + \bar{x}_p\ell_p. \tag{117}$$

Thus the sixth integral vanishes:

$$I_{VI} = \int_{\partial\Omega}\frac{1}{R}\ell_i n_p\ell_p n_q\ell_q dS_y$$
$$= \int_0^{2\pi}\left(\frac{H_0}{R}\right)\ell_i\left(\frac{R}{H_0} + \bar{x}_p\ell_p\right)\left(\frac{R}{H_0} + \bar{x}_q\ell_q\right)d\phi$$
$$= \int_0^{2\pi}\frac{R}{H_0}\ell_i d\phi + 2\bar{x}_p\int_0^{2\pi}\ell_i\ell_p d\phi + \bar{x}_p\bar{x}_q\int_0^{2\pi}\frac{H_0}{R}\ell_i\ell_p\ell_q d\phi$$
$$= -2\pi\bar{x}_i + 2\pi\bar{x}_i = 0. \tag{118}$$

(vii) Finally the last integral is written as

$$I_{VII} = \int_{\partial\Omega}\frac{1}{R}n_i n_j\ell_k n_p\ell_p n_q\ell_q dS_y$$
$$= \int_0^{2\pi}\left(\frac{H_0}{R}\right)\left(\frac{R}{H_0}\ell_i + \bar{x}_i\right)\left(\frac{R}{H_0}\ell_j + \bar{x}_j\right)\ell_k\left(\frac{R}{H_0} + \bar{x}_p\ell_p\right)\left(\frac{R}{H_0} + \bar{x}_q\ell_q\right)d\phi$$
$$= \int_0^{2\pi}\left(\frac{R}{H_0}\right)^3 \ell_i\ell_j\ell_k d\phi + 2\bar{x}_p\int_0^{2\pi}\left(\frac{R}{H_0}\right)^2 \ell_i\ell_j\ell_k\ell_p d\phi + \bar{x}_j\int_0^{2\pi}\left(\frac{R}{H_0}\right)^2 \ell_i\ell_k d\phi$$
$$+ \bar{x}_i\int_0^{2\pi}\left(\frac{R}{H_0}\right)^2 \ell_j\ell_k d\phi + \bar{x}_p\bar{x}_q\int_0^{2\pi}\left(\frac{R}{H_0}\right)\ell_i\ell_j\ell_k\ell_p\ell_q d\phi$$
$$+ 2\bar{x}_p\bar{x}_j\int_0^{2\pi}\left(\frac{R}{H_0}\right)\ell_i\ell_k\ell_p d\phi + 2\bar{x}_p\bar{x}_i\int_0^{2\pi}\left(\frac{R}{H_0}\right)\ell_j\ell_k\ell_p d\phi + \bar{x}_i\bar{x}_j\int_0^{2\pi}\left(\frac{R}{H_0}\right)\ell_k d$$

$$+\bar{x}_j\bar{x}_p\bar{x}_q\int_0^{2\pi}\ell_i\ell_k\ell_p\ell_q d\phi+\bar{x}_i\bar{x}_p\bar{x}_q\int_0^{2\pi}\ell_j\ell_k\ell_p\ell_q d\phi+2\bar{x}_p\bar{x}_i\bar{x}_j\int_0^{2\pi}\ell_p\ell_k d\phi$$

$$+\bar{x}_i\bar{x}_j\bar{x}_p\bar{x}_q\int_0^{2\pi}\left(\frac{H_0}{R}\right)\ell_p\ell_q\ell_k d\phi. \tag{119}$$

All these components have been evaluated before. After lengthy calculation, it finally yields

$$I_{VII}=-\frac{\pi}{2}t\left(1-\frac{t^2}{2}\right)\delta_{ij}r_k+\frac{\pi}{2}t\left(1-t^2\right)(\delta_{ik}r_j+\delta_{jk}r_i)+\frac{\pi}{2}t^3 r_i r_j r_k. \tag{120}$$

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