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Effective elastic moduli of two dimensional solids with distributed cohesive microcracks

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Abstract

Effective elastic properties of a defected solid with distributed cohesive micro-cracks are estimated based on homogenization of the Dugdale–Bilby–Cottrell–Swinden (Dugdale–BCS) type micro-cracks in a two dimensional elastic representative volume element (RVE).

Since the cohesive micro-crack model mimics various realistic bond forces at micro-scale, a statistical average of cohesive defects can effectively represent the overall properties of the material due to bond breaking or crack surface separation in small scale. The newly proposed model is distinctive in the fact that the resulting effective moduli are found to be pressure sensitive.
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1. Introduction

Much progress have been made in the past few decades in the development of macroscopic homogenization of defected composites. Ideally, a successful homogenization scheme should stem from a realistic simulation of failure mechanism at the micro level, so that it can predict material failure in a statistical manner. For example, at micro-level, the failure mechanism due to void growth is well supported by many experimental observations of ductile materials, e.g. McClintock (1968), Budiansky et al. (1982), Duva and Hutchinson (1984), Pardoen and Hutchinson (2000), among others. A statistical homogenization model of void growth is the well-known Gurson model (Gurson, 1977), or its contemporary and computational version, Gurson–Tvergaard–Needleman (GTN) model (Tvergaard, 1981, 1982, 1984), which is the primary phenomenological inelastic damage model in engineering applications.

On the other hand, in most brittle, quasi-brittle, and ductile materials at nano-scale, material's failure mechanism may be attributed to nucleation and coalescence of micro-cracks as well. Although many micro-crack based models have been proposed to describe elastic brittle failure process (e.g., Budiansky and O'Connell, 1976; Hoenig, 1979; Hutchinson, 1987; Lemaitre, 1992; Kachanov, 1992, 1994; Pan and Weng, 1995; Krajcinovic, 1996), few cohesive micro-crack damage models are available for both ductile and quasi-brittle materials, if any.

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Cohesive crack models have long been regarded as a sensible approximation to model fracture, fatigue, and other types of failure phenomena of solids. Since Barenblatt (1959, 1962) and Dugdale's pioneer contribution (Dugdale, 1960), cohesive crack models have been studied extensively. Notable contributions have been made by Bilby, Cottrell and Swinden (1963, 1964), Keer and Mura (1965), Goodier (1968), Rice (1968b), among others.

In reality, in front of the tip of a cohesive crack, there is a very small cohesive zone that is crucial to material failure. How to assess the overall effects attributed from cohesive zone degradation is important for study brittle/ductile fracture at nano- or sub-nano-scale. In this paper, an analytical homogenization procedure is developed to homogenize an elastic solid with randomly distributed cohesive cracks – the Dugdale–BCS cracks. The homogenization leads to a new estimate of effective elastic moduli at macro-level. A new type of pressure sensitive constitutive relations is obtained, which reflects the accumulated damaged effect due to distribution of cohesive micro-cracks.

In Section 2, an averaging theorem is proposed for elastic solids containing randomly distributed cohesive cracks, which provides the theoretical foundation for ensuing homogenization process. In Section 3, the solution of Dugdale–BCS crack is obtained under hydrostatic tension. Effective elastic material properties of a cohesive RVE is presented in Section 4 for both self-consistent scheme and dilute suspension scheme. Section 5 summarizes the distinctive features of the new findings.

2. Averaging theorem

For elastic solids containing cohesive cracks, there is no averaging theorem available in the literature. An extension of averaging theorem for solids containing traction-free defects to solids containing cohesive defects will provide sound theoretical footing for our analysis.

Define macro stress tensor as

$$\Sigma_{\alpha\beta} := \langle \sigma_{\alpha\beta} \rangle = \frac{1}{V} \int_V \sigma_{\alpha\beta} \, \mathrm{d}V, \tag{2.1}$$

where Greek letters range from 1 to 2.

Theorem 2.1. *Suppose*

(1) *A* 2D *elastic representative volume element contains N Dugdale–BCS cracks*;
(2) *The traction on the remote boundary of the RVE are generated by a constant stress tensor, i.e.* $t_\alpha^\infty = \sigma_{\beta\alpha}^\infty n_\beta$, *and* $\sigma_{\alpha\beta}^\infty =$ *const*.

Then average stress of the RVE equals to the remote constant stress, i.e.

$$\langle \sigma_{\alpha\beta} \rangle = \Sigma_{\alpha\beta} = \sigma_{\alpha\beta}^\infty. \tag{2.2}$$

Proof. We first consider the average stress in a 2D representative volume element with a single Dugdale–BCS crack. On the elastic crack surface, ∂V_{ec}, the traction is zero, and inside the cohesive zone, ∂V_{cz}, the cohesive traction is constant. Assume that body force is absent. Using the divergence theorem, one can show that

$$\begin{aligned}\langle \sigma_{\alpha\beta} \rangle &= \frac{1}{V} \int_V \sigma_{\alpha\beta} \, \mathrm{d}V = \frac{1}{V} \int_V (\sigma_{\gamma\beta} x_\alpha)_{,\gamma} \, \mathrm{d}V = \left\{ \int_V \sigma_{\gamma\beta}^\infty \delta_{\alpha\gamma} \, \mathrm{d}V - \int_{\partial V_{ec}} 0 \cdot x_\alpha n_\gamma \, \mathrm{d}S - \int_{\partial V_{cz}} \sigma_{\gamma\beta} x_\alpha n_\gamma \, \mathrm{d}S \right\} \\ &= \sigma_{\alpha\beta}^\infty - \frac{1}{V} \int_{\partial V_{cz}} t_\beta x_\alpha \, \mathrm{d}S,\end{aligned} \tag{2.3}$$

where $t_\beta := \sigma_{\beta\alpha} n_\alpha$ is the constant cohesive traction. Note that $\partial V_{cz} = \partial V_{cz+} \cup \partial V_{cz-}$ and $|\partial V_{cz+}| = |\partial V_{cz-}| = 1/2 |\partial V_{cz}|$. Thus, the last term in (2.3) vanishes, i.e.

$$\frac{1}{V} \int_{\partial V_{cz}} t_\beta x_\alpha \, \mathrm{d}S = \frac{1}{V} \left(\int_{\partial V_{cz+}} t_\beta^+ x_\alpha \, \mathrm{d}S + \int_{\partial V_{cz-}} t_\beta^- x_\alpha \, \mathrm{d}S \right) = \frac{1}{2V} (t_\beta^+ + t_\beta^-) x_\alpha |\partial V_{cz}| = 0 \tag{2.4}$$

because $t_\beta^+ = -t_\beta^-$. Therefore $\langle \sigma_{\alpha\beta} \rangle = \sigma_{\alpha\beta}^\infty$. For an RVE with N cohesive macrocracks, it can be shown that

$$\langle \sigma_{\alpha\beta} \rangle = \sigma_{\alpha\beta}^\infty - \frac{1}{V} \sum_{k=1}^N \int_{\partial V_{czk}} t_\beta^{(k)} x_\alpha \, \mathrm{d}S. \tag{2.5}$$

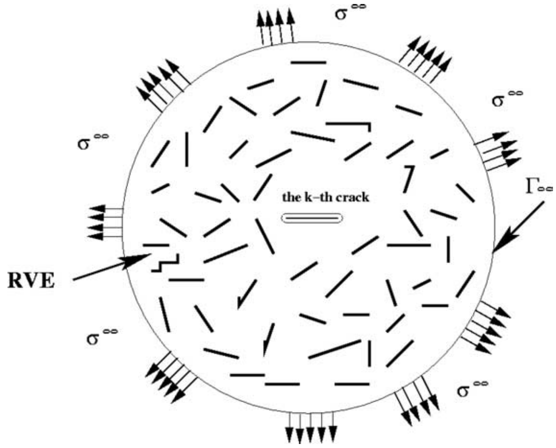

Fig. 1. A two-dimensional RVE with randomly distributed cracks.

Following the similar argument, one can show that the last term vanishes. We then conclude that

$$\langle\sigma_{\alpha\beta}\rangle = \Sigma_{\alpha\beta} = \sigma^{\infty}_{\alpha\beta}. \qquad \square \tag{2.6}$$

3. Solution of Dugdale–BCS crack under hydrostatic tension

Before proceeding to homogenization, we first briefly outline the mode I Dugdale–BCS crack solution in a representative volume element (RVE). The stress path is designed to be monotonic hydrostatic tension to maintain purely mode I condition at all crack tips with different orientations. Presence of mode II and mode III conditions is avoided by this specification (see Li and Morgan, 2003 for related discussions on the case of model III cohesive cracks).

Consider a two-dimensional RVE with a Dugdale–BCS crack in the center. Hydrostatic tension is applied on the remote boundary of the RVE, Γ_∞.

$$\Sigma_{11} = \Sigma_{22} = \sigma^\infty, \quad \forall \mathbf{x} \in \Gamma_\infty. \tag{3.1}$$

On macro-level, the remote stress σ^∞ may be related with the spherical stress Σ_m of the RVE

$$\Sigma_m = \phi\sigma^\infty = \begin{cases} \dfrac{2}{3}\sigma^\infty, & \text{plane stress;} \\ \dfrac{2}{3}(1+\nu^*)\sigma^\infty, & \text{plane strain.} \end{cases} \tag{3.2}$$

The complete Dugdale–BCS mode I crack solution can be obtained via superposition of the trivial solution and the crack solution:

(i) *Trivial solution.* Consider an RVE without cracks. A trivial solution of hydrostatic stress state is: $\forall \mathbf{x} \in V$,

$$\sigma^{(0)}_{11} = \sigma^\infty, \qquad \sigma^{(0)}_{22} = \sigma^\infty, \qquad \sigma^{(0)}_{12} = 0 \tag{3.3}$$

and

$$\begin{bmatrix} \epsilon^{(0)}_{11} \\ \epsilon^{(0)}_{22} \\ 2\epsilon^{(0)}_{12} \end{bmatrix} = \frac{1}{\mu^*} \begin{bmatrix} \frac{\kappa^*+1}{8} & \frac{\kappa^*-3}{8} & 0 \\ \frac{\kappa^*-3}{8} & \frac{\kappa^*+1}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma^\infty \\ \sigma^\infty \\ 0 \end{bmatrix}, \tag{3.4}$$

where superscript (0) indicating trivial solution, and material constants μ^*, κ^*, or ν^*, depend on ensuing homogenization procedures, and κ^* is Kolosov's constant

$$\kappa^* := \begin{cases} \dfrac{3-\nu^*}{1+\nu^*}, & \text{plane stress;} \\ 3-4\nu^*, & \text{plane strain.} \end{cases} \tag{3.5}$$

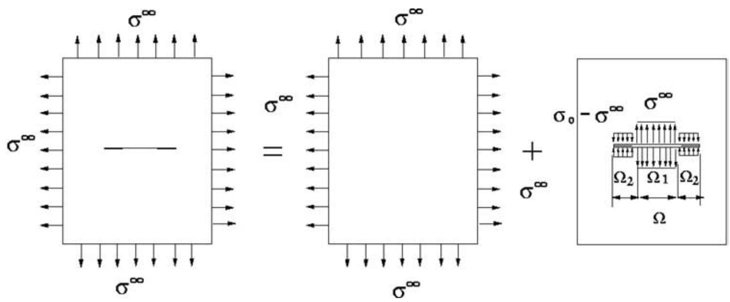

Fig. 2. Illustration of superposition of cohesive crack problem.

By inspection, one may find the displacement fields of the trivial solution

$$u_1^{(0)} = \frac{\kappa^*+1}{8\mu^*}\sigma^{\infty}x_1 + \frac{\kappa^*-3}{8\mu^*}\sigma^{\infty}x_1 = \frac{\kappa^*-1}{4\mu^*}\sigma^{\infty}x_1, \tag{3.6}$$

$$u_2^{(0)} = \frac{\kappa^*-3}{8\mu^*}\sigma^{\infty}x_2 + \frac{\kappa^*+1}{8\mu^*}\sigma^{\infty}x_2 = \frac{\kappa^*-1}{4\mu^*}\sigma^{\infty}x_2. \tag{3.7}$$

(ii) *Crack solution.* The crack solution has to satisfy the remote boundary conditions

$$\sigma_{11}^{(c)} = \sigma_{22}^{(c)} = \sigma_{12}^{(c)} = 0, \qquad r = \sqrt{x_1^2 + x_2^2} \to \infty \tag{3.8}$$

and crack surface traction boundary conditions and symmetric condition

$$\sigma_{22}^{(c)} = -\sigma^{\infty}, \qquad \forall x_2 = 0,\ |x_1| < a, \tag{3.9}$$

$$\sigma_{22}^{(c)} = \sigma_0 - \sigma^{\infty}, \quad \forall x_2 = 0,\ a \leqslant |x_1| < b, \tag{3.10}$$

$$u_2^{(c)} = 0, \qquad \forall\ x_2 = 0,\ |x_1| > b, \tag{3.11}$$

where σ_0 is the material's cohesive strength, i.e., the onset normal stress value prior to surface separation. Here the superscript (c) indicates the crack solution.

The stress field solution on x_1 axis ($x_2 = 0$) is well known (e.g., Mura, 1987, pages 280–285)

$$\sigma_{11}^{(c)}(x_1,0) = \sigma_{22}^{(c)}(x_1,0) = -\frac{\mathrm{d}}{\mathrm{d}x_1}\left\{\int_0^b \frac{q(t)H(x_1-t)}{\sqrt{x_1^2-t^2}}\,\mathrm{d}t\right\}, \tag{3.12}$$

where $H(\cdot)$ is the Heaviside function, and

$$q(t) = \begin{cases} \sigma^{\infty}t, & t < a, \\ \sigma^{\infty}t - \dfrac{2}{\pi}\sigma_0\cos^{-1}\left(\dfrac{a}{t}\right)t, & a \leqslant t < b. \end{cases} \tag{3.13}$$

The stress distribution along x_1 axis are:

(1) $\forall 0 < |x_1| < a$,

$$\sigma_{11}^{(c)}(x_1,0) = \sigma_{22}^{(c)}(x_1,0) = -\sigma^{\infty}, \qquad \sigma_{12}^{(c)} = 0;$$

(2) $\forall a \leqslant |x_1| < b$,

$$\sigma_{11}^{(c)}(x_1,0) = \sigma_{22}^{(c)}(x_1,0) = -\sigma^{\infty} + \sigma_0, \qquad \sigma_{12}^{(c)}(x_1,0) = 0;$$

(3) $\forall b < |x_1| < \infty$,

$$\begin{aligned}\sigma_{11}^{(c)}(x_1,0) &= \sigma_{22}^{(c)}(x_1,0) \\ &= -\sigma^{\infty} + \sigma_0 + \frac{\sigma_0 a}{\pi}\frac{\mathrm{d}}{\mathrm{d}x_1}\left\{\frac{x_1}{a}\sin^{-1}\left[\frac{x_1^2(b^2-2a^2)+a^2b^2}{b^2(x_1^2-a^2)}\right] + \sin^{-1}\left[\frac{x_1^2+a^2-2b^2}{x_1^2-a^2}\right]\right\},\end{aligned}$$

$$\sigma_{12}^{(c)}(x_1,0) = 0.$$

Inside the cohesive zone ($a < |x_1| < b$), the total stress (superscript (t)) components are,

$$\sigma_{11}^{(t)} = \sigma_{11}^{(0)} + \sigma_{11}^{(c)} = \sigma_0, \tag{3.14}$$

$$\sigma_{22}^{(t)} = \sigma_{22}^{(0)} + \sigma_{22}^{(c)} = \sigma_0, \tag{3.15}$$

$$\sigma_{12}^{(t)} = 0. \tag{3.16}$$

It is assumed that the microscopic yielding of the material is governed by the Huber–von Mises criterion. Since the shear stresses are all zero inside the cohesive zone,

$$\sqrt{\frac{1}{2}\big((\sigma_{11}^{(t)} - \sigma_{22}^{(t)})^2 + (\sigma_{22}^{(t)} - \sigma_{33}^{(t)})^2 + (\sigma_{33}^{(t)} - \sigma_{11}^{(t)})^2\big)} = \sigma_Y \tag{3.17}$$

which links the cohesive strength with the uniaxial yield stress of the virgin material,

$$\sigma_Y = \chi\sigma_0 = \begin{cases} \sigma_0, & \text{plane stress,} \\ (1-2\nu^*)\sigma_0, & \text{plane strain.} \end{cases} \tag{3.18}$$

In the crack solution, displacement fields along the x_1 axis are

$$u_1^{(c)}(x_1, \pm 0) = \frac{1-\kappa^*}{4\mu^*}\sigma^\infty x_1 + \frac{\kappa^*-1}{4\mu^*}\sigma_0 \begin{cases} x_1 + a, & \forall -b < x_1 \leqslant -a, \\ 0, & \forall -a < x_1 < a, \\ x_1 - a, & \forall a \leqslant x_1 < b \end{cases} \tag{3.19}$$

and

$$u_2^{(c)}(x_1, \pm 0) = \pm\frac{1+\kappa^*}{4\pi\mu^*}\sigma_0 \cdot \left\{ x_1 \ln\left| \frac{x_1\sqrt{b^2-a^2} - a\sqrt{b^2-x_1^2}}{x_1\sqrt{b^2-a^2} + a\sqrt{b^2-x_1^2}} \right| - a\ln\left| \frac{\sqrt{b^2-a^2} - \sqrt{b^2-x_1^2}}{\sqrt{b^2-a^2} + \sqrt{b^2-x_1^2}} \right| \right\}. \tag{3.20}$$

Therefore, inside the cohesive zone ($a < |x_1| < b$),

$$u_1^{(t)}(x_1, 0) = \frac{\kappa^*-1}{4\mu^*}\sigma_0 \begin{cases} x_1 + a, & \forall -b < x_1 \leqslant -a, \\ 0, & \forall -a < x_1 \leqslant a, \\ x_1 - a, & \forall a \leqslant x_1 < b \end{cases} \tag{3.21}$$

and $u_2^{(t)}(x_1, 0) = u_2^{(c)}(x_1, 0)$. The elastic crack opening volume can be expressed as

$$V(a) = \int_{-a}^{a} [u_2^{(t)}]\,\mathrm{d}x_1 = \frac{1+\kappa^*}{2\mu^*}a^2\sigma_0\left\{ \left(1 - \frac{\sigma^\infty}{\sigma_0}\right)\tan\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right) - \frac{4}{\pi}\ln\left[\cos\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]\right\}, \tag{3.22}$$

and the total crack opening volume as

$$V(b) = \int_{-b}^{b} [u_2^{(t)}]\,\mathrm{d}x_1 = \frac{1+\kappa^*}{2\mu^*}\sigma_0 a^2 \tan\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right). \tag{3.23}$$

The crack tip opening displacement is given by Rice (1968a),

$$\delta_t = u_2^{(t)}(a, +0) - u_2^{(t)}(a, -0) = \frac{(1+\kappa^*)\sigma_0 a}{\pi\mu^*}\ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]. \tag{3.24}$$

As shown by Rice (1968a, 1968b), the value of J-integral of mode I Dugdale–BCS crack is

$$J = \sigma_0\delta_t = \frac{1+\kappa^*}{\pi\mu^*}\sigma_0^2 a \ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]. \tag{3.25}$$

Since J is related to energy release, assume $\frac{\partial}{\partial\ell}\mathcal{R}_1 = J$, where $\ell = 2a$ is the total length of the crack. It may be found that

$$\mathcal{R}_1 = \frac{1+\kappa^*}{\pi\mu^*}\sigma_0^2 a^2 \ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]. \tag{3.26}$$

Note that in nonlinear fracture mechanics, J may not be the exact surface separation energy release rate (see Wnuk, 1990).

The total energy release can be calculated as

$$\mathcal{R}_2=\sigma^\infty V(b)-\sigma_0\big(V(b)-V(a)\big)=\frac{2(1+\kappa^*)}{\pi\mu^*}\sigma_0^2a^2\ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]. \tag{3.27}$$

Considering the fact that $\mathcal{R}_2=2\mathcal{R}_1$, one can then combine the two into a single form

$$\mathcal{R}_\omega=\frac{\omega(1+\kappa^*)}{\pi\mu^*}\sigma_0^2a^2\ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right],\quad \omega=1,2. \tag{3.28}$$

4. Effective elastic material properties of a cohesive RVE

An efficient procedure of finding effective elastic material properties of a damaged solid is energy method. By balance of strain energy density of a damage process, one may be able to find effective potential energy density and hence effective complementary energy density, which are assumed to be potential functions of average strain and average stress. Consider an RVE with N randomly distributed cohesive cracks, the effective potential energy density in an RVE can be calculated by evaluating average energy release rate.

$$\overline{W}=\langle\sigma_{\alpha\beta}\epsilon_{\alpha\beta}\rangle-W^c-\frac{\mathcal{R}_\omega}{V},\quad \omega=1,2. \tag{4.1}$$

Define the crack density

$$f:=\sum_{k=1}^{N}\frac{\pi a_k^2}{V}. \tag{4.2}$$

By the Legender transform, the effective complementary energy density can be written as

$$\overline{W}^c=W^c+\frac{\mathcal{R}_\omega}{V}=\frac{1}{2}D_{\alpha\beta\zeta\eta}\Sigma_{\alpha\beta}\Sigma_{\zeta\eta}+\left(\sum_{k=1}^{N}\frac{\pi a_k^2}{V}\right)\frac{\omega\sigma_0^2(1+\kappa^*)}{\pi^2\mu^*}\ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]. \tag{4.3}$$

Since $\overline{W}^c$ is a potential function of $\Sigma_{\alpha\beta}$ and by the averaging Theorem 2.2 ($\Sigma_{\alpha\beta}=\sigma^\infty_{\alpha\beta}$), one may find

$$\mathcal{E}_{\alpha\beta}=\frac{\partial\overline{W}^c}{\partial\Sigma_{\alpha\beta}}=\epsilon^{(0)}_{\alpha\beta}+\epsilon^{(\mathrm{add})}_{\alpha\beta}=D_{\alpha\beta\zeta\eta}\Sigma_{\zeta\eta}+\frac{\partial}{\partial\sigma^\infty}\left(\frac{\omega\sigma_0^2(1+\kappa^*)f}{\pi^2\mu^*}\ln\left[\sec\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]\right)\frac{\partial\sigma^\infty}{\partial\Sigma_{\alpha\beta}}, \tag{4.4}$$

where $\epsilon^0_{\alpha\beta}:=D_{\alpha\beta\zeta\eta}\Sigma_{\zeta\eta}$, $\mathcal{E}_{\alpha\beta}=\overline{D}_{\alpha\beta\zeta\eta}\Sigma_{\zeta\eta}$. Since $\Sigma_{\alpha\beta}=\sigma^\infty\delta_{\alpha\beta}$, we have

$$\epsilon^{(\mathrm{add})}_{\alpha\beta}=\frac{\omega(1+\kappa^*)}{4\pi\mu^*}f\left[\frac{\sigma_0}{\sigma^\infty}\tan\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]\sigma^\infty\delta_{\alpha\beta},\quad \omega=1,2. \tag{4.5}$$

For an in-plane homogenization, the effective material properties may become transversely isotropic after homogenization. As an approximation, we assume that after homogenization the damaged medium is still isotropic both macroscopically and microscopically (inside the RVE).

The essence of self-consistent method is to consider the effect of micro-crack interaction (Hill, 1965a, 1965b, 1967; Budiansky and O'Connell, 1976). Let $\kappa^*=\bar\kappa,\mu^*=\bar\mu$ in Eqs. (4.4) and (4.5). We want to find a global isotropic tensor $\mathbf{H}=\frac{h_1}{2}\mathbf{1}^{(2)}\otimes\mathbf{1}^{(2)}+h_2\mathbf{1}^{(4s)}$, such that

$$\boldsymbol{\epsilon}^{(\mathrm{add})}=\mathbf{H}:\boldsymbol{\Sigma}, \tag{4.6}$$

where $\mathbf{1}^{(2)}=\delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ is the second identity tensor, and $\mathbf{1}^{(4s)}=\frac{1}{2}(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_\ell$ is the fourth order symmetric identity tensor.

By utilizing additional strain formula (4.6), one can determine the effective compliance tensor,

$$\overline{\mathbf{D}}=\mathbf{D}+\mathbf{H}. \tag{4.7}$$

Apparently, the information carried in (4.5) is not sufficient to determine the $\mathbf{H}$ tensor, because it does not contain the information on degradation of shear modulus. To evaluate $\mathbf{H}$, an additional provision on averaging is needed.

Let

$$\mathbf{D} = \frac{1}{3K}\mathbf{E}^{(1)} + \frac{1}{2\mu}\mathbf{E}^{(2)}, \tag{4.8}$$

$$\overline{\mathbf{D}} = \frac{1}{3\bar{K}}\mathbf{E}^{(1)} + \frac{1}{2\bar{\mu}}\mathbf{E}^{(2)}, \tag{4.9}$$

$$\mathbf{H} = (h_1 + h_2)\mathbf{E}^{(1)} + h_2\mathbf{E}^{(2)}, \tag{4.10}$$

where $\mu = \frac{E}{2(1+\nu)}$ and

$$\frac{1}{3K} = \begin{cases} \dfrac{1-\nu}{E}, & \text{plane stress;} \\ \dfrac{1-\nu-2\nu^2}{E}, & \text{plane strain,} \end{cases} \tag{4.11}$$

and similar expressions hold for $\bar{\mu}$ and $\overline{K}$. Note that

$$\mathbf{E}^{(1)} := \frac{1}{2}\delta_{\alpha\beta}\delta_{\zeta\eta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\zeta \otimes \mathbf{e}_\eta, \tag{4.12}$$

$$\mathbf{E}^{(2)} := \frac{1}{2}(\delta_{\alpha\zeta}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\zeta} - \delta_{\alpha\beta}\delta_{\zeta\eta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\zeta \otimes \mathbf{e}_\eta. \tag{4.13}$$

Under hydrostatic remote loading, Eq. (4.6) is only valid when $\boldsymbol{\Sigma} = \sigma^\infty \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$. It only admits one scalar equation,

$$\overline{\mathbf{D}} : (\sigma^\infty \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta) = (\mathbf{D} + \mathbf{H}) : (\sigma^\infty \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta). \tag{4.14}$$

Combining Eqs. (4.5) and (4.14), one may find that

$$\frac{1}{3\overline{K}} = \frac{1}{3K} + (h_1 + h_2) = \frac{1}{3K} + \left(\frac{1}{3\overline{K}} + \frac{1}{2\bar{\mu}}\right)\omega f\left[\frac{\sigma_0}{\pi\sigma^\infty}\tan\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right] \tag{4.15}$$

by virtue of identities $\mathbf{E}^{(1)} : \mathbf{1}^{(2)} = \mathbf{1}^{(2)}$ and $\mathbf{E}^{(2)} : \mathbf{1}^{(2)} = \mathbf{0}$.

In Eq. (4.15), there are two unknowns, $\overline{K}$ and $\bar{\mu}$, or equivalently h_1 and h_2. An additional condition is needed to uniquely determine $\overline{\mathbf{D}}$ or $\mathbf{H}$. Impose the restriction

$$\frac{\overline{K}}{K} = \frac{\bar{\mu}}{\mu}. \tag{4.16}$$

This implies that the relative reduction of the bulk modulus is the same as that of the shear modulus. This restriction will guarantee the positive definiteness of the effective strain energy, and it is reasonable for hydrostatic loading condition.

Remark 4.1. There may be some other possibilities for additional restriction. For example, $h_1 = 0$. However, in this particular problem, the restriction, $h_1 = 0$, may not guarantee the positive definiteness of the effective strain energy.

A direct consequence of (4.16) is

$$\bar{\nu} = \nu. \tag{4.17}$$

Then for plane problems, one may find

$$\frac{\overline{K}}{K} = \frac{\bar{\mu}}{\mu} = \begin{cases} 1 - \dfrac{2\omega f}{1-\nu}\left[\left(\dfrac{\sigma_0}{\pi\sigma^\infty}\right)\tan\left(\dfrac{\pi\sigma^\infty}{2\sigma_0}\right)\right], & \text{plane stress;} \quad \text{(a)} \\ 1 - \dfrac{2(1-\nu^2)\omega f}{1-\nu-2\nu^2}\left[\left(\dfrac{\sigma_0}{\pi\sigma^\infty}\right)\tan\left(\dfrac{\pi\sigma^\infty}{2\sigma_0}\right)\right], & \text{plane strain.} \quad \text{(b)} \end{cases} \tag{4.18}$$

For plane stress problems, the effective shear modulus via self consistent scheme, $\bar{\mu}/\mu$, is plotted for various loading intensities by choosing $\omega = 2, \nu = 0.1$. From Fig. 3, effective modulus decreases linearly with respect to f. With increasing loading intensity σ^∞/σ_0, the slope of degradation line is increasing until material breaks down even with infinitesimal amount of defects at the limit of $\sigma^\infty \to \sigma_0$. Another limit case, $\sigma^\infty/\sigma_0 = 0$ corresponds to elastic crack limit, where cohesive strength approaches infinity.

By chosing $K^* = K$ and $\mu^* = \mu$, one may find that the effective shear modulus via dilute distribution approach is

$$\frac{\bar{\mu}}{\mu} = \left\{1 + \frac{\omega(1+\kappa)f}{4}\left[\left(\frac{\sigma_0}{\pi\sigma^\infty}\right)\tan\left(\frac{\pi\sigma^\infty}{2\sigma_0}\right)\right]\right\}^{-1}, \tag{4.19}$$

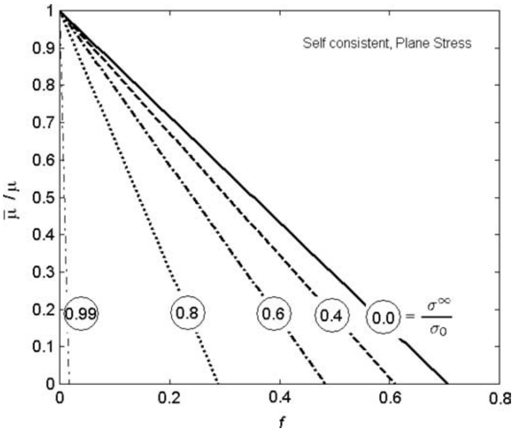

Fig. 3. Effective modulus for various loading intensities (self-consistent).

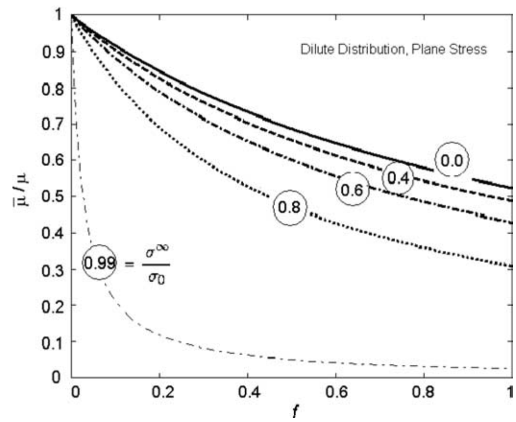

Fig. 4. Effective modulus for various loading intensities (dilute distribution).

where κ is Kolosov's constant (see (3.5)).

The effective shear modulus is plotted in Fig. 4 for various loading intensities with same set of parameters.

5. Concluding remarks

The most distinguished features of new cohesive crack damage models are:

(1) The effective macro-constitutive relations are different from the micro-constitutive relations: After homogenization, the effective elastic material properties are pressure sensitive, whereas at micro-level, the elastic stiffness and compliance tensors are constant;
(2) Self-consistent homogenization scheme yields a linear degradation of effective elastic moduli with respect to crack density f, and it predicts a critical value of crack density, f_c, at which the material completely loses its strength;
(3) Both self-consistent and dilute suspension homogenization schemes predict pressure sensitive effective material response. When the ratio of hydrostatic stress and the true yield reaches a finite value, i.e.

$$\frac{\sigma^\infty}{\sigma_0} \to 1 \quad \Rightarrow \quad \frac{\Sigma_m}{\sigma_Y} \to \begin{cases} \frac{2}{3}, & \text{plane stress,} \\ \frac{2(1+\nu)}{3(1-2\nu)}, & \text{plane strain} \end{cases} \tag{5.1}$$

a complete failure in material will be captured with even an infinitesimal amount of initial damage, if the loading stress reaches to the material's theoretical strength, whereas under the same condition, the Gurson model will not predict a complete material failure unless the hydrostatic stress approaches infinity.
(4) The key to accurately calculate effective elastic properties is to determine the energy release contribution to the material damage process. The energy release in nonlinear fracture mechanical process is consumed in several different dissipation processes, e.g. surface separation, heat generation, dislocation movement, and may be even phase transformation, etc. In fact, Kfouri (1979) and Wnuk (1990) have studied energy release caused by crack extension of two-dimensional Dugdale–BCS cracks. To avoid complications, monotonic hydrostatic tension load is prescribed in our analysis to maintain purely mode I crack condition. An in-depth study may be needed to refine the analysis proposed here.

A three-dimensional cohesive crack damage model has been derived recently by the present authors (Li and Wang, 2004) as well.

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